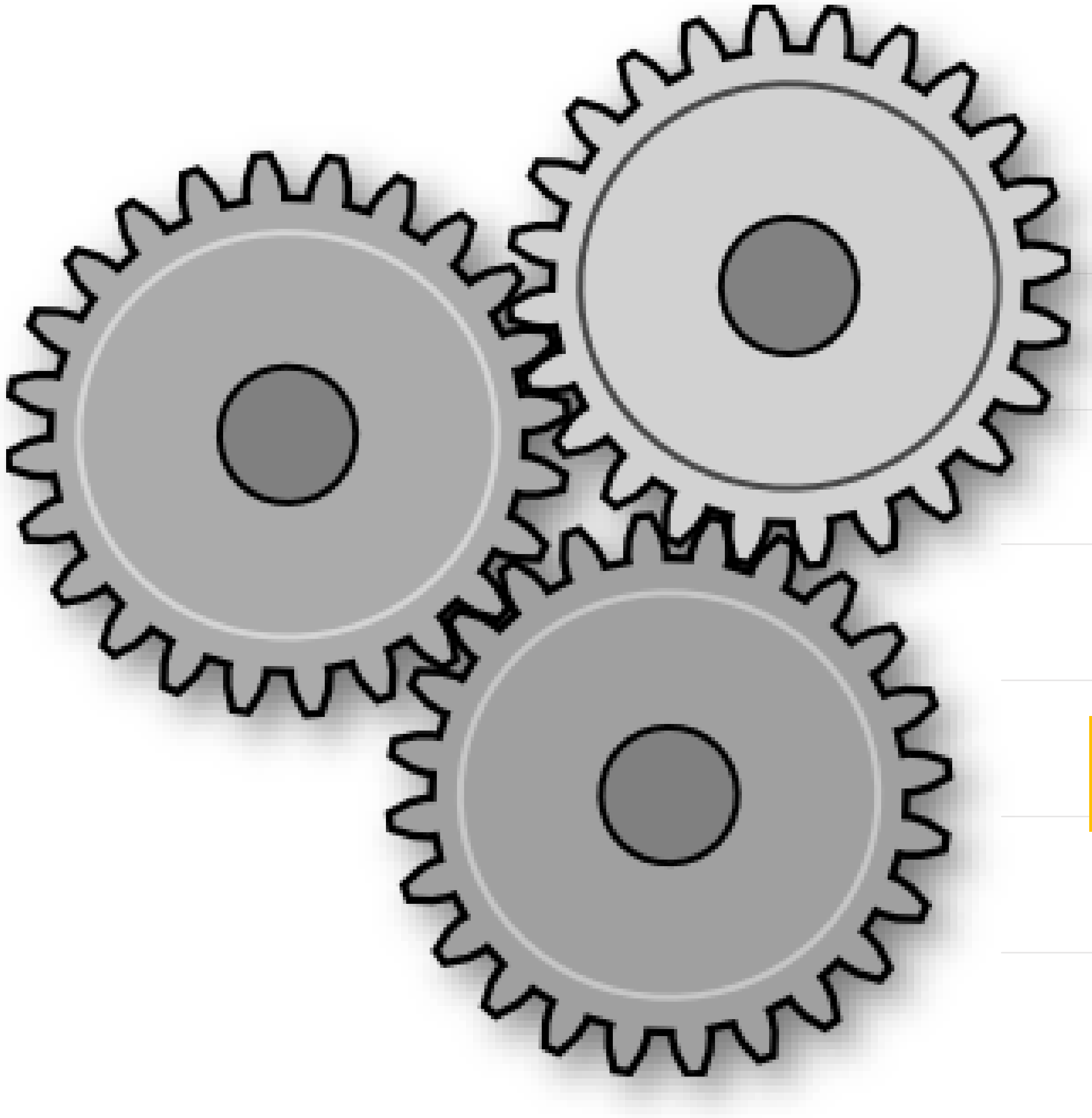
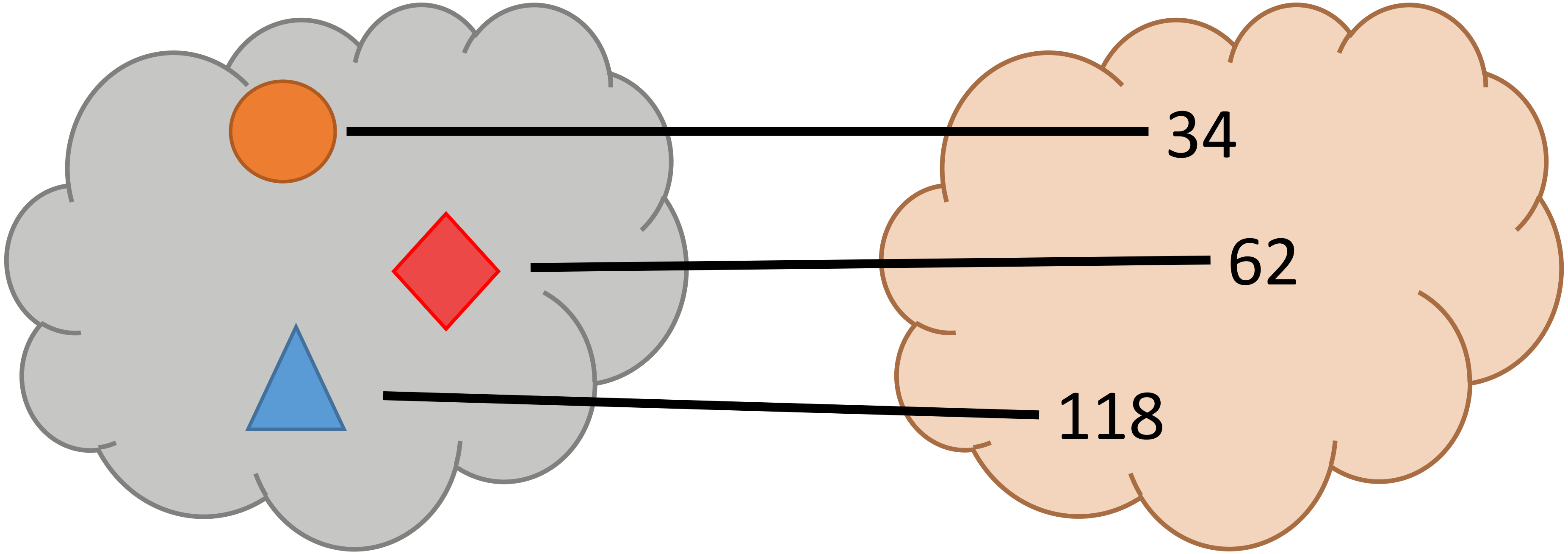


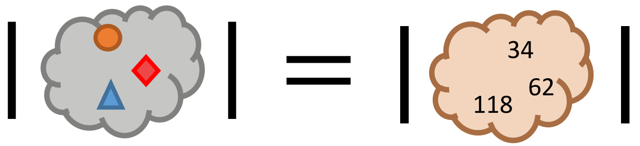
Undecidable Problems

Analiza Algoritmilor

Image source: https://www.wpclipart.com/signs_symbol/optical_illusions/impossible_objects/impossible_gears.png.html

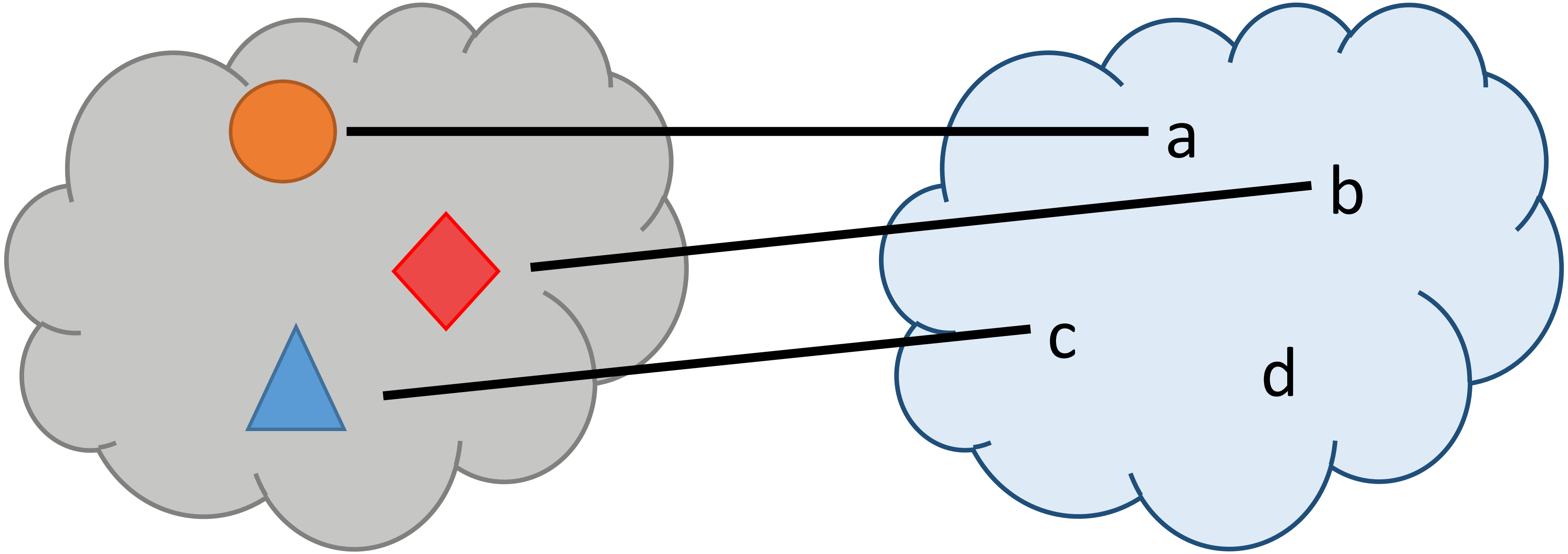
Cardinality

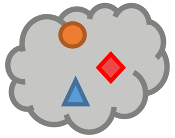
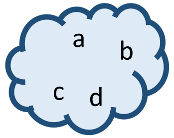




Two sets *have the same cardinality* $\stackrel{\text{def}}{=}$ there exists a ***bijection*** between them

Cardinality (2)



|  | $\leq$ |  |

A has lower cardinality than $\mathrm{B} \stackrel{\text{def}}{=}$ there exists an ***injection*** from A to B

Cardinality of infinite sets – even naturals

$\mathbb{N}$	$\{ 2k \mid k \in \mathbb{N}\}$
0	0
1	2
2	4
3	6
4	8
5	10
6	12
7	14
8	16
9	18
…	…

$$|\mathbb{N}| = |\{ 2k \mid k \in \mathbb{N}\}|$$

Cardinality of infinite sets – integers

$\mathbb{N}$	$\mathbb{Z}$
0	0
1	-1
2	1
3	-2
4	2
5	-3
6	3
7	-4
8	4
9	-5
…	…

$$|\mathbb{N}| = |\mathbb{Z}|$$

Cardinality of the binary strings

$\{0,1\}^*$	$\mathbb{N}$
ε	0
0	1
1	2
00	3
01	4
10	5
11	6
000	7
001	8
…	…

$$|\{0,1\}^*| = |\mathbb{N}|$$

Cardinality of Turing Machine encodings

	g_1	g_2	g_3
q_1	q_1, g_1, d_2	q_1, g_2, d_2	q_5, g_3, d_1
q_5	q_2, g_1, d_3	q_3, g_2, d_3	q_3, g_3, d_3

enc(isEven) = 0101010100 11 010010100100 11 01000100000100010 11 0000010100101000 11
000001001000100100 11 000001000100010001000

$$|\mathbb{M}| \leq |\{0,1\}^*|$$

Cardinality of $[0, 1)$

$\mathbb{N}$	$[0, 1)$
0	0.**2**169460714693289171222729971066…
1	0.7**6**31763176317631763176317631763…
2	0.68**9**5404337592950674809357136346…
3	0.689**5**404337592950600000000000000…
4	0.3805**4**54783500225038873577418048…
5	0.92007**7**8257579442881690521804839…
6	0.313769**2**127249575025762653280831…
7	0.3700368**0**70995432261445179661871…
8	0.24107515**5**7594412258050924569386…
…	

https://www.smbc-comics.com/comic/how-math-works

Cardinality of $[0, 1)$

$\mathbb{N}$	$[0, 1)$
0	0.2169460714693289171222729971066…
1	0.7631763176317631763176317631763…
2	0.6895404337592950674809357136346…
3	0.6895404337592950600000000000000…
4	0.3805454783500225038873577418048…
5	0.9200778257579442881690521804839…
6	0.3137692127249575025762653280831…
7	0.3700368070995432261445179661871…
8	0.2410751557594412258050924569386…
…	

0.269547205…

0.122222122…

$$|\mathbb{N}| \neq |[0, 1)|$$

https://www.smbc-comics.com/comic/how-math-works

Representing arbitrary decision problems

ε	$f(\varepsilon) = TRUE$	1
0	$f(0) = FALSE$	0
1	$f(1) = TRUE$	1
00	$f(00) = TRUE$	1
01	$f(01) = FALSE$	0
10	$f(10) = TRUE$	1
11	$f(11) = FALSE$	0
000	$f(000) = FALSE$	0
...	$\ldots$	...

10110100...

Cardinality of decision problems

$\mathbb{N}$	$\mathbb{D}$
0	**0**101010111011010101001001000100...
1	0**1**10001001100110000100011100100...
2	01**0**0101001000101111110100100010...
3	111**1**111011110111100001010001001...
4	1001**1**00101011010110001000100001...
5	10010**0**0111001000010010101101100...
6	001010**0**100100111000100001000101...
7	1100100**0**10001000010001000100001...
8	11101101**0**0010110001110001001000...
...	

Cardinality of decision problems

$\mathbb{N}$	$\mathbb{D}$
0	0101010111011010101001001000100...
1	0110001001100110000100011100100...
2	0100101001000101111110100100010...
3	1111111011110111100001010001001...
4	1001100101011010110001000100001...
5	1001000111001000010010101101100...
6	0010100100100111000100001000101...
7	1100100010001000010001000100001...
8	1110110100010110001110001001000...
...	

010110000...

101001111...

$$|\mathbb{N}| \neq |\mathbb{D}|$$

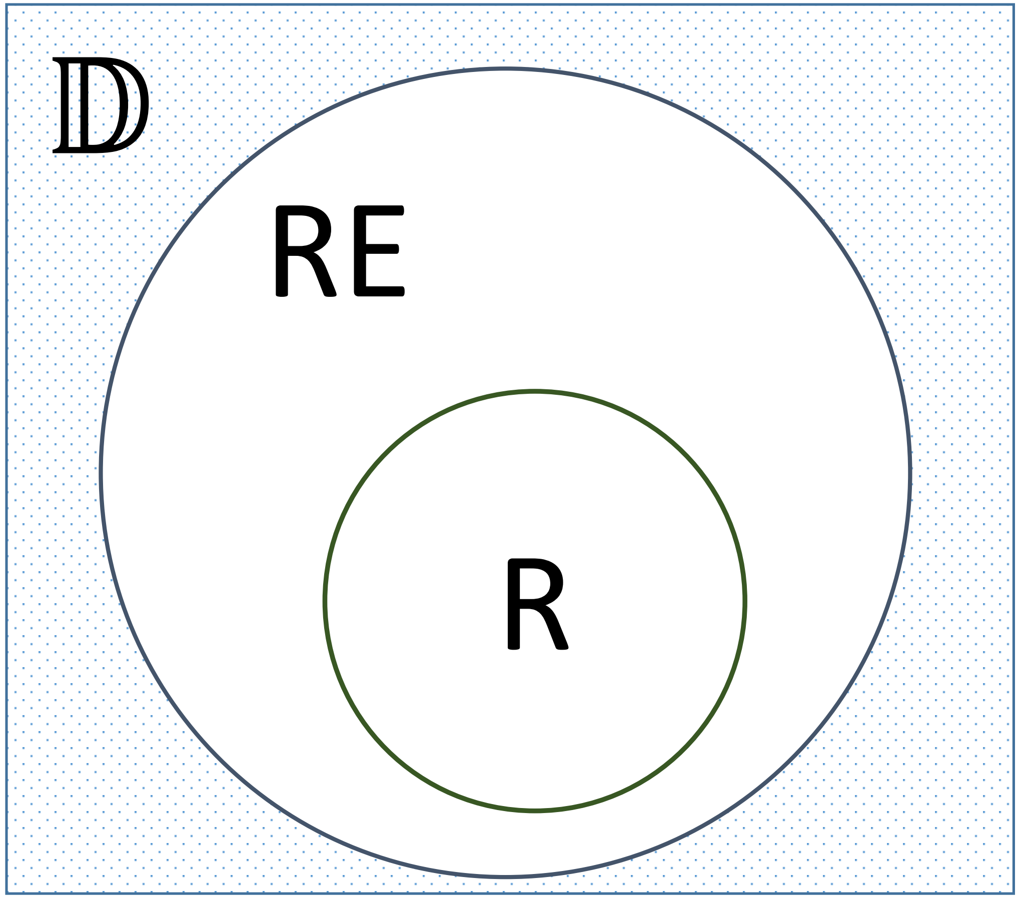
Cardinality of decision problems (2)

$\mathbb{N}$	$\mathbb{D}$
0	0000000000000000000000000000000…
1	1000000000000000000000000000000…
2	1100000000000000000000000000000…
3	1110000000000000000000000000000…
4	1111000000000000000000000000000…
5	1111100000000000000000000000000…
6	1111110000000000000000000000000…
7	1111111000000000000000000000000…
8	111111110000000000000000000000…
…	

$$|\mathbb{N}| \leq |\mathbb{D}|$$

(and unacceptable)

Conclusion: there exist undecidable problems!



Cardinality of R

	w_0	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	...
M_0	0	1	0	1	0	1	0	1	1	...
M_1	0	1	1	0	0	0	1	0	0	...
M_2	0	1	0	0	1	0	1	0	0	...
M_3	1	1	1	1	1	1	1	0	1	...
M_4	1	0	0	1	1	0	0	1	0	...
M_5	1	0	0	1	0	0	0	1	1	...
M_6	0	0	1	0	1	0	0	1	0	...
...	...									

$f_0(w_0) = FALSE$

$$|R| = |\mathbb{M}| = |\{0,1\}^*| = |\mathbb{N}|$$

Cardinality of R

	w_0	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	...
M_0	0	1	0	1	0	1	0	1	1	...
M_1	0	1	1	0	0	0	1	0	0	...
M_2	0	1	0	0	1	0	1	0	0	...
M_3	1	1	1	1	1	1	1	0	1	...
M_4	1	0	0	1	1	0	0	1	0	...
M_5	1	0	0	1	0	0	0	1	1	...
M_6	0	0	1	0	1	0	0	1	0	...
...	...									

$f_0(w_0) = FALSE$

$f_4(w_7) = TRUE$

$$|R| = |\mathbb{M}| = |\{0,1\}^*| = |\mathbb{N}|$$

Cardinality of R

	w_0	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	…
M_0	0	1	0	1	0	1	0	1	1	…
M_1	0	1	1	0	0	0	1	0	0	…
M_2	0	1	0	0	1	0	1	0	0	…
M_3	1	1	1	1	1	1	1	0	1	…
M_4	1	0	0	1	1	0	0	1	0	…
M_5	1	0	0	1	0	0	0	1	1	…
M_6	0	0	1	0	1	0	0	1	0	…
…	…									

$f_0(w_0) = FALSE$

$f_4(w_7) = TRUE$

$$f_d(w_i) = \overline{f_i(w_i)}$$

Cardinality of R

	w_0	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	…
M_0	0	1	0	1	0	1	0	1	1	…
M_1	0	1	1	0	0	0	1	0	0	…
M_2	0	1	0	0	1	0	1	0	0	…
M_3	1	1	1	1	1	1	1	0	1	…
M_4	1	0	0	1	1	0	0	1	0	…
M_5	1	0	0	1	0	0	0	1	1	…
M_6	0	0	1	0	1	0	0	1	0	…
…	…									

$f_0(w_0) = FALSE$

$f_4(w_7) = TRUE$

$$f_d \notin R$$

$$f_d(w_i) = \overline{f_i(w_i)}$$

The Halting Problem

$$\mathrm{HALT}\big(enc(M,w)\big) = \begin{cases} TRUE & M[w]\ halts \\ FALSE & otherwise \end{cases}$$

The Halting Problem – placement