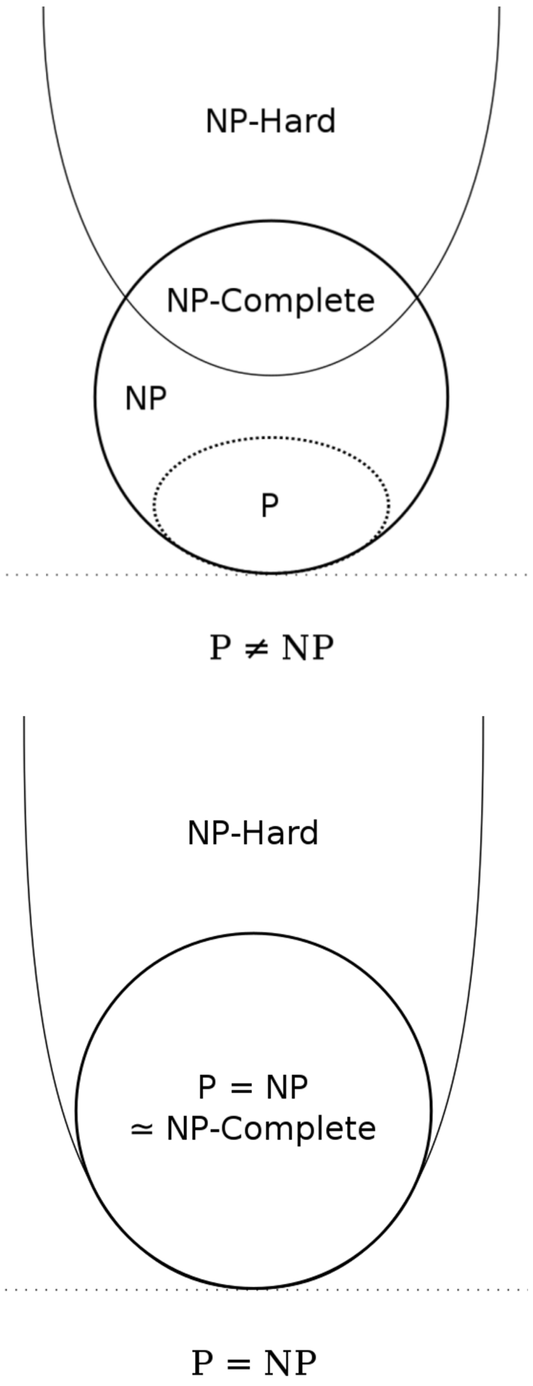


12

Hardness and Completeness

Analiza Algoritmilor

“Hardness”

$$f \leq_m g$$

g is “harder” than f

If we can decide g, then we can decide f

“Polynomial hardness”

$$f \leq_P g$$

g is “polynomially harder” than f

If we can decide g in time $T(n)$,

then we can decide f in time $n^k + T(n^k)$

NP-Hardness: $g \in \boldsymbol{NPH} \stackrel{\text{def}}{=} \forall f \in \boldsymbol{NP}, f \leq_P g$

NP-Completeness: $g \in \boldsymbol{NPC} \stackrel{\text{def}}{=} \begin{cases} g \in \boldsymbol{NPH} \\ g \in \boldsymbol{NP} \end{cases}$

The Cook-Levin Theorem

Stephen Cook (1939 -)

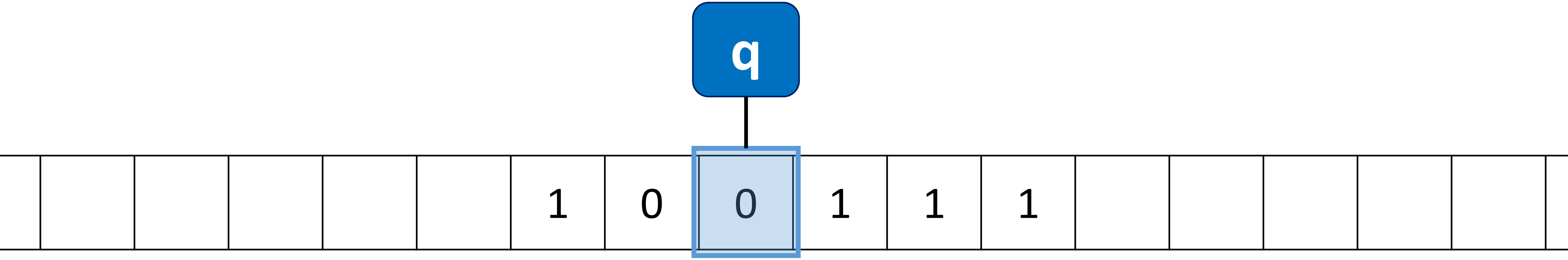
“SAT is NP-Complete”

Leonid Levin (1948 -)

Cook: https://commons.wikimedia.org/wiki/File:Prof.Cook.jpg

Levin: https://commons.wikimedia.org/wiki/File:LeonidLevin2010.jpg

Configuration strings



10q0111

Computation history strings

$q110111\#1q10111\#11q0111\#110q111\#1101q11\#11011q1\#$

$110111q \square \#11011p1\#1101p10\#110p100\#11p0000\#11H1000$

$|w|^k \times |w|^k$ tableau of $N[110100101]$

$|w|^k$

#	q_1	1	1	0	1	0	0	1	0	1									#
#	0	q_1	1	0	1	0	0	1	0	1									#
#	0	0	q_1	0	1	0	0	1	0	1									#
#	0	0	1	q_1	1	0	0	1	0	1									#
#	0	0	1	0	q_1	0	0	1	0	1									#
#	0	0	1	0	1	1	0	1	0	1	1	0	q_3	1	0	1			#
#	0	0	1	0	1	1	0	1	0	1	1	0	Y	1	0	1			#
#	0	0	1	0	1	1	0	1	0	1	1	0	Y	1	0	1			#
#	0	0	1	0	1	1	0	1	0	1	1	0	Y	1	0	1			#

$|w|^k$

$$x_{1,1,\#} \wedge \overline{x_{1,1,0}} \wedge \overline{x_{1,1,1}} \wedge \overline{x_{1,1,\square}} \wedge \overline{x_{1,1,q_1}} \wedge \overline{x_{1,1,q_2}} \ldots \wedge \overline{x_{1,1,N}} \wedge \overline{x_{1,1,Y}}$$

#	q_1	1	1	0	1	0	0	1	0	1									#
#	0	q_1	1	0	1	0	0	1	0	1									#
#	0	0	q_1	0	1	0	0	1	0	1									#
#	0	0	1	q_1	1	0	0	1	0	1									#
#	0	0	1	0	q_1	0	0	1	0	1									#
#	0	0	1	0	1	1	0	1	0	1	1	0	q_3	1	0	1			#
#	0	0	1	0	1	1	0	1	0	1	1	0	Y	1	0	1			#
#	0	0	1	0	1	1	0	1	0	1	1	0	Y	1	0	1			#
#	0	0	1	0	1	1	0	1	0	1	1	0	Y	1	0	1			#

$$x_{1,1,\#} \wedge \overline{x_{1,1,0}} \wedge \overline{x_{1,1,1}} \wedge \overline{x_{1,1,\square}} \wedge \overline{x_{1,1,\mathrm{q}_1}} \wedge \overline{x_{1,1,\mathrm{q}_2}} \ldots \wedge \overline{x_{1,1,N}} \wedge \overline{x_{1,1,Y}}$$

$$x_{4,5,q_1} \wedge \overline{x_{4,5,0}} \wedge \overline{x_{4,5,1}} \wedge \overline{x_{4,5,\square}} \wedge \overline{x_{1,1,\#}} \wedge \overline{x_{1,1,\mathrm{q}_2}} \ldots \wedge \overline{x_{1,1,N}} \wedge \overline{x_{1,1,Y}}$$

#	q_1	1	1	0	1	0	0	1	0	1									#
#	0	q_1	1	0	1	0	0	1	0	1									#
#	0	0	q_1	0	1	0	0	1	0	1									#
#	0	0	1	q_1	1	0	0	1	0	1									#
#	0	0	1	0	q_1	0	0	1	0	1									#
#	0	0	1	0	1	1	0	1	0	1	1	0	q_3	1	0	1			#
#	0	0	1	0	1	1	0	1	0	1	1	0	Y	1	0	1			#
#	0	0	1	0	1	1	0	1	0	1	1	0	Y	1	0	1			#
#	0	0	1	0	1	1	0	1	0	1	1	0	Y	1	0	1			#

$x_{14,15,0} \vee x_{14,15,1} \vee x_{14,15,\square} \vee x_{14,15,\#} \vee x_{14,15,\mathrm{q}_1} \vee x_{14,15,q_2} \vee \ldots \vee x_{14,15,N} \vee x_{14,15,Y}$

$\wedge \left(\overline{x_{14,15,0}} \vee \overline{x_{14,15,1}}\right) \wedge \left(\overline{x_{14,15,0}} \vee \overline{x_{14,15,\square}}\right) \wedge \left(\overline{x_{14,15,0}} \vee \overline{x_{14,15,\#}}\right) \wedge \cdots \wedge \left(\overline{x_{14,15,0}} \vee \overline{x_{14,15,\mathrm{Y}}}\right)$

$\wedge \left(\overline{x_{14,15,Y}} \vee \overline{x_{14,15,0}}\right) \wedge \left(\overline{x_{14,15,Y}} \vee \overline{x_{14,15,1}}\right) \wedge \left(\overline{x_{14,15,Y}} \vee \overline{x_{14,15,\square}}\right) \wedge \cdots \wedge \left(\overline{x_{14,15,Y}} \vee \overline{x_{14,15,\mathrm{N}}}\right)$

...

$\wedge \left(\overline{x_{14,15,Y}} \vee \overline{x_{14,15,0}}\right) \wedge \left(\overline{x_{14,15,Y}} \vee \overline{x_{14,15,1}}\right) \wedge \left(\overline{x_{14,15,Y}} \vee \overline{x_{14,15,\square}}\right) \wedge \cdots \wedge \left(\overline{x_{14,15,Y}} \vee \overline{x_{14,15,\mathrm{N}}}\right)$

#	q_1	1	1	0	1	0	0	1	0	1									#
#	0	q_1	1	0	1	0	0	1	0	1									#
#	0	0	q_1	0	1	0	0	1	0	1									#
#	0	0	1	q_1	1	0	0	1	0	1									#
#	0	0	1	0	q_1	0	0	1	0	1									#
#	0	0	1	0	1	1	0	1	0	1	1	0	q_3	1	0	1			#
#	0	0	1	0	1	1	0	1	0	1	1	0	Y	1	0	1			#
#	0	0	1	0	1	1	0	1	0	1	1	0	Y	1	0	1			#
#	0	0	1	0	1	1	0	1	0	1	1	0	Y	1	0	1			#

$$\phi_{\text{cell}} = \bigwedge_{1 \le i,j \le |w|^k} \left[\left(\bigvee_{s \in \Pi} x_{i,j,s} \right) \wedge \left(\bigwedge_{\substack{s,t \in \Pi \\ s \neq t}} \left(\overline{x_{i,j,s}} \vee \overline{x_{i,j,t}} \right) \right) \right]$$

#	q_1	1	1	0	1	0	0	1	0	1									#
#	0	q_1	1	0	1	0	0	1	0	1									#
#	0	0	q_1	0	1	0	0	1	0	1									#
#	0	0	1	q_1	1	0	0	1	0	1									#
#	0	0	1	0	q_1	0	0	1	0	1									#
#	0	0	1	0	1	1	0	1	0	1	1	0	q_3	1	0	1			#
#	0	0	1	0	1	1	0	1	0	1	1	0	Y	1	0	1			#
#	0	0	1	0	1	1	0	1	0	1	1	0	Y	1	0	1			#
#	0	0	1	0	1	1	0	1	0	1	1	0	Y	1	0	1			#

$$\phi_{init} = x_{1,1,\#} \wedge x_{1,2,q_1} \wedge \left(\bigwedge_{1 \le i \le |w|} x_{1,2+i,w_i} \right) \wedge \left(\bigwedge_{|w|+3 \le j < |w|^k} x_{1,j,\square} \right) \wedge x_{1,|w|^k,\#}$$

#	q_1	1	1	0	1	0	0	1	0	1									#
#	0	q_1	1	0	1	0	0	1	0	1									#
#	0	0	q_1	0	1	0	0	1	0	1									#
#	0	0	1	q_1	1	0	0	1	0	1									#
#	0	0	1	0	q_1	0	0	1	0	1									#
#	0	0	1	0	1	1	0	1	0	1	1	0	q_3	1	0	1			#
#	0	0	1	0	1	1	0	1	0	1	1	0	Y	1	0	1			#
#	0	0	1	0	1	1	0	1	0	1	1	0	Y	1	0	1			#
#	0	0	1	0	1	1	0	1	0	1	1	0	Y	1	0	1			#

(isEven’, 100) pairs

$\#$	0	1	$\square$	$\#$	$\#$
$\#q_1 100\#$	0	1	$\square$	$\#$	$\square\#$

pair 1

$q_1 0$	$q_1 1$	$q_1 \square$	$q_2 0$	$q_2 1$	$0 q_2 \square$	$1 q_2 \square$	$q_3 0$	$q_3 1$	$q_3 \square$
$0 q_2$	$1 q_2$	$1N$	$0 q_2$	$1 q_2$	$q_3 0$	$q_3 1$	$0Y$	$1 q_3$	$1 q_3$

$0Y$	$1Y$	$\square Y$	$Y0$	$Y1$	$Y\square$	$0N$	$1N$	$\square N$	$N0$	$N1$	$N\square$	$Y\#\#$	$N\#\#$
Y	Y	Y	Y	Y	Y	N	N	N	N	N	N	$\#$	$\#$

#	q_1	1	1	0	1	0	0	1	0	1									#
#	0	q_1	1	0	1	0	0	1	0	1									#
#	0	0	q_1	0	1	0	0	1	0	1									#
#	0	0	1	q_1	1	0	0	1	0	1									#
#	0	0	1	0	q_1	0	0	1	0	1									#
#	0	0	1	0	1	1	0	1	0	1	1	0	q_3	1	0	1			#
#	0	0	1	0	1	1	0	1	0	1	1	0	Y	1	0	1			#
#	0	0	1	0	1	1	0	1	0	1	1	0	Y	1	0	1			#
#	0	0	1	0	1	1	0	1	0	1	1	0	Y	1	0	1			#

#	q_1	1	1	0	1	0	0	1	0	1									#
#	0	q_1	1	0	1	0	0	1	0	1									#
#	0	0	q_1	0	1	0	0	1	0	1									#
#	0	0	1	q_1	1	0	0	1	0	1									#
#	0	0	1	0	q_1	0	0	1	0	1									#
#	0	0	1	0	1	1	0	1	0	1	1	0	q_3	1	0	1			#
#	0	0	1	0	1	1	0	1	0	1	1	0	Y	1	0	1			#
#	0	0	1	0	1	1	0	1	0	1	1	0	Y	1	0	1			#
#	0	0	1	0	1	1	0	1	0	1	1	0	Y	1	0	1			#

#	q_1	1	1	0	1	0	0	1	0	1									#
#	0	q_1	1	0	1	0	0	1	0	1									#
#	0	0	q_1	0	1	0	0	1	0	1									#
#	0	0	1	q_1	1	0	0	1	0	1									#
#	0	0	1	0	q_1	0	0	1	0	1									#
#	0	0	1	0	1	1	0	1	0	1	1	0	q_3	1	0	1			#
#	0	0	1	0	1	1	0	1	0	1	1	0	Y	1	0	1			#
#	0	0	1	0	1	1	0	1	0	1	1	0	Y	1	0	1			#
#	0	0	1	0	1	1	0	1	0	1	1	0	Y	1	0	1			#

#	q_1	1	1	0	1	0	0	1	0	1									#
#	0	q_1	1	0	1	0	0	1	0	1									#
#	0	0	q_1	0	1	0	0	1	0	1									#
#	0	0	1	q_1	1	0	0	1	0	1									#
#	0	0	1	0	q_1	0	0	1	0	1									#
#	0	0	1	0	1	1	0	1	0	1	1	0	q_3	1	0	1			#
#	0	0	1	0	1	1	0	1	0	1	1	0	Y	1	0	1			#
#	0	0	1	0	1	1	0	1	0	1	1	0	Y	1	0	1			#
#	0	0	1	0	1	1	0	1	0	1	1	0	Y	1	0	1			#

#	q_1	1	1	0	1	0	0	1	0	1									#
#	0	q_1	1	0	1	0	0	1	0	1									#
#	0	0	q_1	0	1	0	0	1	0	1									#
#	0	0	1	q_1	1	0	0	1	0	1									#
#	0	0	1	0	q_1	0	0	1	0	1									#
#	0	0	1	0	1	1	0	1	0	1	1	0	q_3	1	0	1			#
#	0	0	1	0	1	1	0	1	0	1	1	0	Y	1	0	1			#
#	0	0	1	0	1	1	0	1	0	1	1	0	Y	1	0	1			#
#	0	0	1	0	1	1	0	1	0	1	1	0	Y	1	0	1			#

#	q_1	1	1	0	1	0	0	1	0	1									#
#	0	q_1	1	0	1	0	0	1	0	1									#
#	0	0	q_1	0	1	0	0	1	0	1									#
#	0	0	1	q_1	1	0	0	1	0	1									#
#	0	0	1	0	q_1	0	0	1	0	1									#
#	0	0	1	0	1	1	0	1	0	1	1	0	q_3	1	0	1			#
#	0	0	1	0	1	1	0	1	0	1	1	0	Y	1	0	1			#
#	0	0	1	0	1	1	0	1	0	1	1	0	Y	1	0	1			#
#	0	0	1	0	1	1	0	1	0	1	1	0	Y	1	0	1			#

$$\phi_{transition} = \bigwedge_{1 \leq i < |w|^k, 1 \leq j < |w|^k - 1} window_{i,j}$$

#	q_1	1	1	0	1	0	0	1	0	1									#
#	0	q_1	1	0	1	0	0	1	0	1									#
#	0	0	q_1	0	1	0	0	1	0	1									#
#	0	0	1	q_1	1	0	0	1	0	1									#
#	0	0	1	0	q_1	0	0	1	0	1									#
#	0	0	1	0	1	1	0	1	0	1	1	0	q_3	1	0	1			#
#	0	0	1	0	1	1	0	1	0	1	1	0	Y	1	0	1			#
#	0	0	1	0	1	1	0	1	0	1	1	0	Y	1	0	1			#
#	0	0	1	0	1	1	0	1	0	1	1	0	Y	1	0	1			#

Y?

#	q_1	1	1	0	1	0	0	1	0	1									#
#	0	q_1	1	0	1	0	0	1	0	1									#
#	0	0	q_1	0	1	0	0	1	0	1									#
#	0	0	1	q_1	1	0	0	1	0	1									#
#	0	0	1	0	q_1	0	0	1	0	1									#
#	0	0	1	0	1	1	0	1	0	1	1	0	q_3	1	0	1			#
#	0	0	1	0	1	1	0	1	0	1	1	0	Y	1	0	1			#
#	0	0	1	0	1	1	0	1	0	1	1	0	Y	1	0	1			#
#	0	0	1	0	1	1	0	1	0	1	1	0	Y	1	0	1			#

Y?

#	q_1	1	1	0	1	0	0	1	0	1									#
#	0	q_1	1	0	1	0	0	1	0	1									#
#	0	0	q_1	0	1	0	0	1	0	1									#
#	0	0	1	q_1	1	0	0	1	0	1									#
#	0	0	1	0	q_1	0	0	1	0	1									#
#	0	0	1	0	1	1	0	1	0	1	1	0	q_3	1	0	1			#
#	0	0	1	0	1	1	0	1	0	1	1	0	Y	1	0	1			#
#	0	0	1	0	1	1	0	1	0	1	1	0	Y	1	0	1			#
#	0	0	1	0	1	1	0	1	0	1	1	0	Y	1	0	1			#

Y?

#	q_1	1	1	0	1	0	0	1	0	1									#
#	0	q_1	1	0	1	0	0	1	0	1									#
#	0	0	q_1	0	1	0	0	1	0	1									#
#	0	0	1	q_1	1	0	0	1	0	1									#
#	0	0	1	0	q_1	0	0	1	0	1									#
#	0	0	1	0	1	1	0	1	0	1	1	0	q_3	1	0	1			#
#	0	0	1	0	1	1	0	1	0	1	1	0	Y	1	0	1			#
#	0	0	1	0	1	1	0	1	0	1	1	0	Y	1	0	1			#
#	0	0	1	0	1	1	0	1	0	1	1	0	Y	1	0	1			#

Y?

#	q_1	1	1	0	1	0	0	1	0	1									#
#	0	q_1	1	0	1	0	0	1	0	1									#
#	0	0	q_1	0	1	0	0	1	0	1									#
#	0	0	1	q_1	1	0	0	1	0	1									#
#	0	0	1	0	q_1	0	0	1	0	1									#
#	0	0	1	0	1	1	0	1	0	1	1	0	q_3	1	0	1			#
#	0	0	1	0	1	1	0	1	0	1	1	0	Y	1	0	1			#
#	0	0	1	0	1	1	0	1	0	1	1	0	Y	1	0	1			#
#	0	0	1	0	1	1	0	1	0	1	1	0	Y	1	0	1			#

$$\phi_{accept} = \bigvee_{1 \leq i,j \leq |w|^k} x_{i,j,Y}$$

#	q_1	1	1	0	1	0	0	1	0	1									#
#	0	q_1	1	0	1	0	0	1	0	1									#
#	0	0	q_1	0	1	0	0	1	0	1									#
#	0	0	1	q_1	1	0	0	1	0	1									#
#	0	0	1	0	q_1	0	0	1	0	1									#
#	0	0	1	0	1	1	0	1	0	1	1	0	q_3	1	0	1			#
#	0	0	1	0	1	1	0	1	0	1	1	0	Y	1	0	1			#
#	0	0	1	0	1	1	0	1	0	1	1	0	Y	1	0	1			#
#	0	0	1	0	1	1	0	1	0	1	1	0	Y	1	0	1			#

$$\phi = \phi_{cell} \wedge \phi_{init} \wedge \phi_{transition} \wedge \phi_{accept}$$

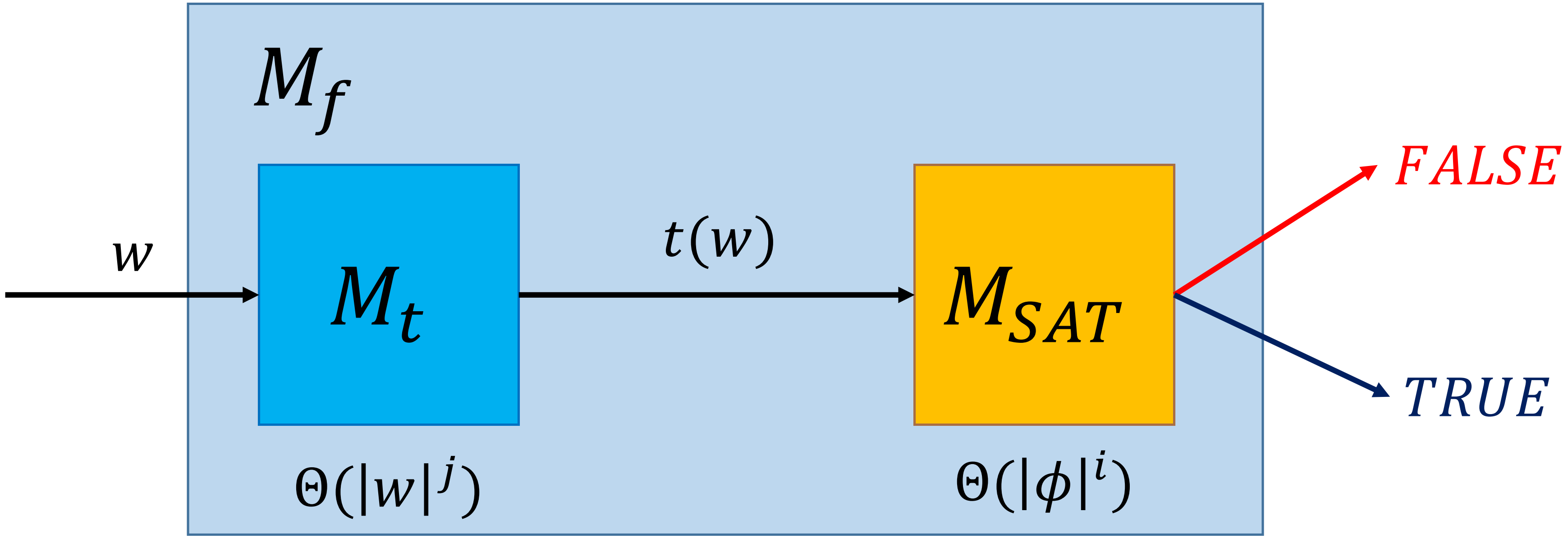
#	q_1	1	1	0	1	0	0	1	0	1									#
#	0	q_1	1	0	1	0	0	1	0	1									#
#	0	0	q_1	0	1	0	0	1	0	1									#
#	0	0	1	q_1	1	0	0	1	0	1									#
#	0	0	1	0	q_1	0	0	1	0	1									#
#	0	0	1	0	1	1	0	1	0	1	1	0	q_3	1	0	1			#
#	0	0	1	0	1	1	0	1	0	1	1	0	Y	1	0	1			#
#	0	0	1	0	1	1	0	1	0	1	1	0	Y	1	0	1			#
#	0	0	1	0	1	1	0	1	0	1	1	0	Y	1	0	1			#

$$\phi = \phi_{cell} \wedge \phi_{init} \wedge \phi_{transition} \wedge \phi_{accept}$$

#	q_1	1	1	0	1	0	0	1	0	1									#
#	0	q_1	1	0	1	0	0	1	0	1									#
#	0	0	q_1	0	1	0	0	1	0	1									#
#	0	0	1	q_1	1	0	0	1	0	1									#
#	0	0	1	0	q_1	0	0	1	0	1									#
#	0	0	1	0	1	1	0	1	0	1	1	0	q_3	1	0	1			#
#	0	0	1	0	1	1	0	1	0	1	1	0	Y	1	0	1			#
#	0	0	1	0	1	1	0	1	0	1	1	0	Y	1	0	1			#
#	0	0	1	0	1	1	0	1	0	1	1	0	Y	1	0	1			#

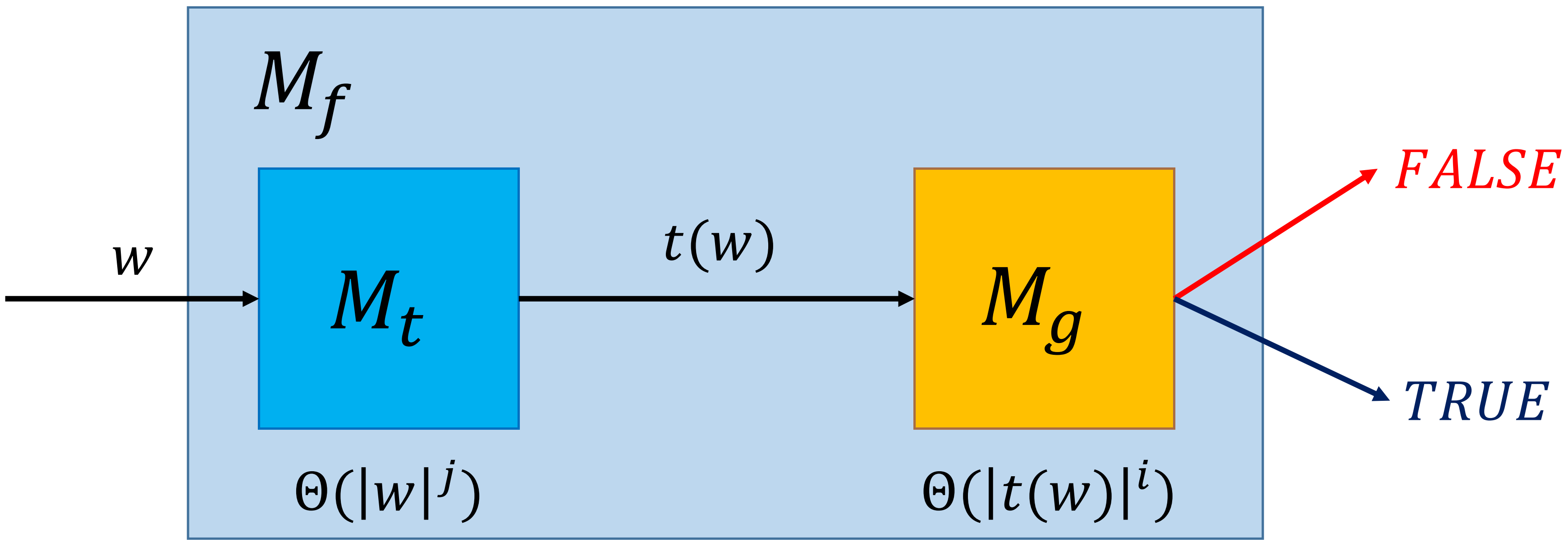
SAT $\in$ ***NPC***

SAT $\in \boldsymbol{P} \Leftrightarrow \boldsymbol{P} = \boldsymbol{NP}$



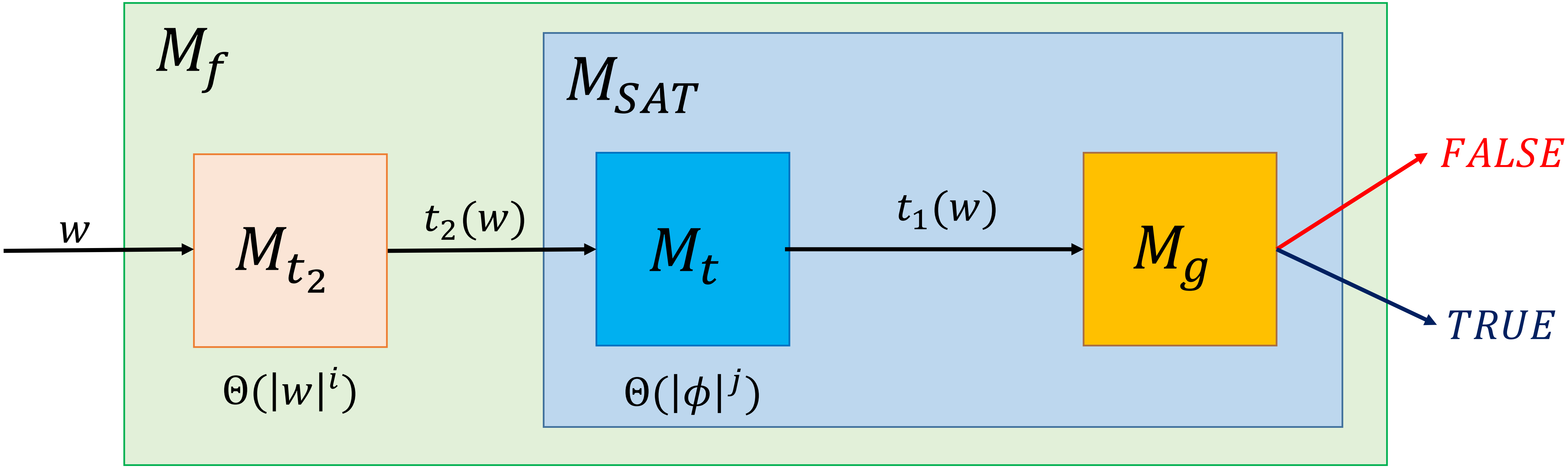
$$\Theta(|w|^{ji})$$

$$(g \in \boldsymbol{NPH} \wedge g \in \boldsymbol{P}) \Leftrightarrow \boldsymbol{P} = \boldsymbol{NP}$$

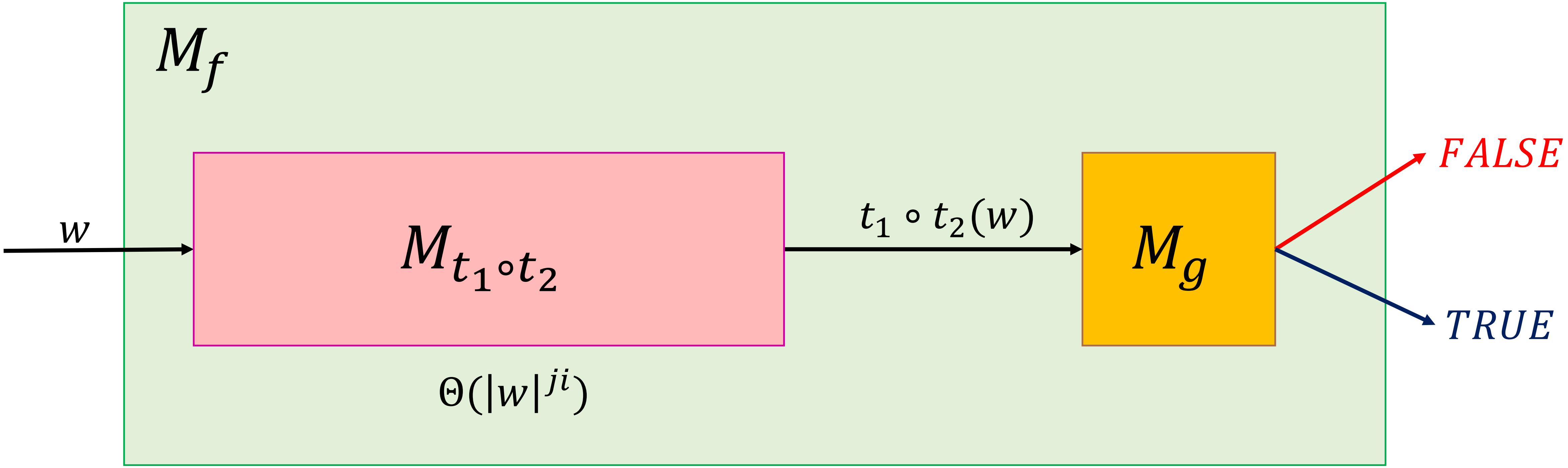


$$\Theta(|w|^{ji})$$

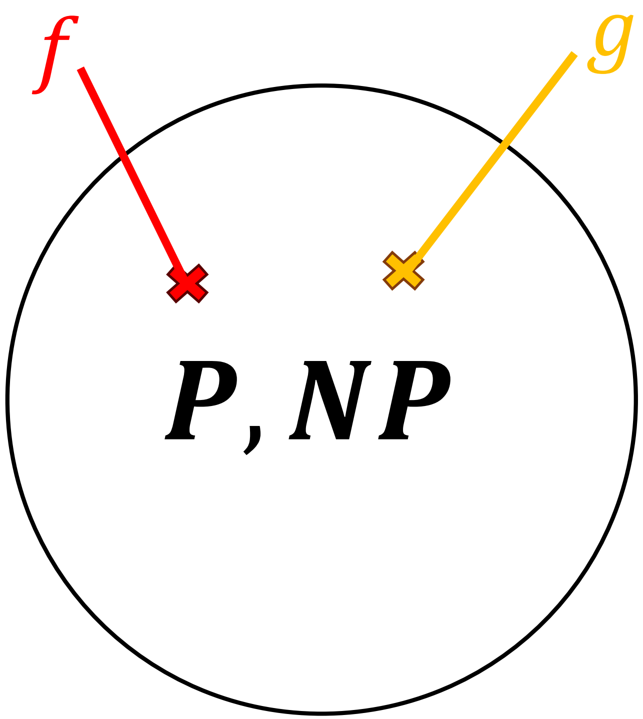
$\text{SAT} \leq_P g \Leftrightarrow g \in \boldsymbol{NPH}$



$\text{SAT} \le_P g \Leftrightarrow g \in \boldsymbol{NPH}$



$\boldsymbol{P = NP \Leftrightarrow NP \simeq NPC}$

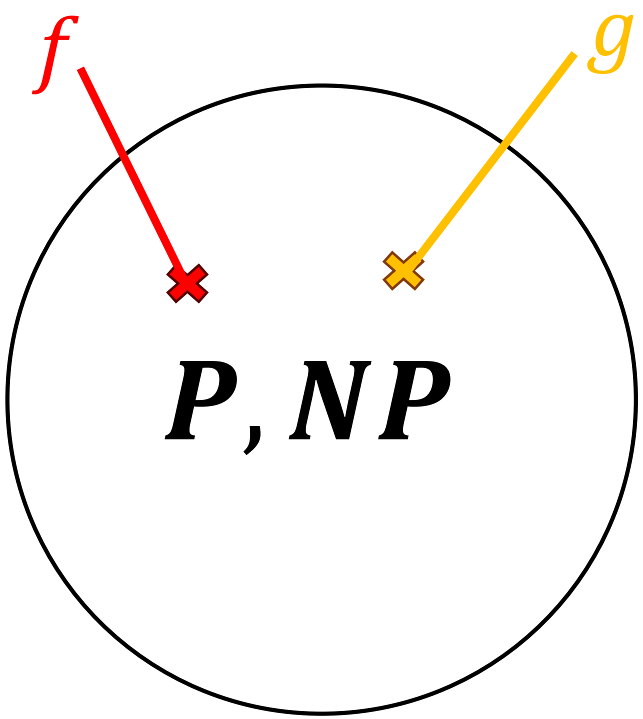


$$f \leq_P g$$

$M_t[w]$:

1. run $D_f[w]$
2. if $D_f[w] \to \text{TRUE}$:
3. output w_T (s.t. $g(w_T) = TRUE$)
4. else:
5. output w_F (s.t. $g(w_F) = FALSE$)

$P = NP \Leftrightarrow NP \simeq NPC$



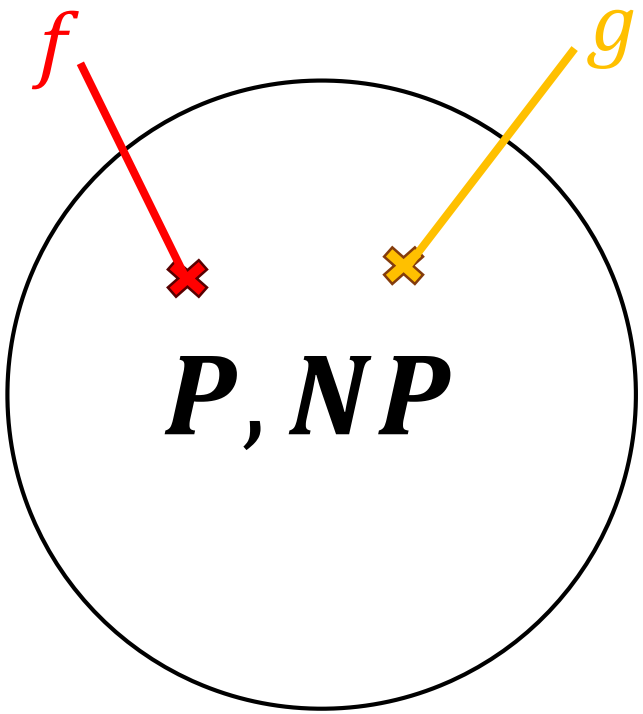
$$f \leq_P g$$

$M_t[w]$:

1. run $D_f[w]$
2. if $D_f[w] \rightarrow \text{TRUE}$:
3. output w_T (s.t. $g(w_T) = TRUE$)
4. else:
5. output w_F (s.t. $g(w_F) = FALSE$)

There must exist such w_T and w_F

$$P = NP \Leftrightarrow NP = NPC \setminus \{f_{TRUE}, f_{FALSE}\}$$



$$f \leq_P g$$

$M_t[w]$:

1. run $D_f[w]$
2. if $D_f[w] \rightarrow$ TRUE:
3. output w_T (s.t. $g(w_T) = TRUE$)
4. else:
5. output w_F (s.t. $g(w_F) = FALSE$)

$$g \neq f_{TRUE} \vee g \neq f_{FALSE}$$

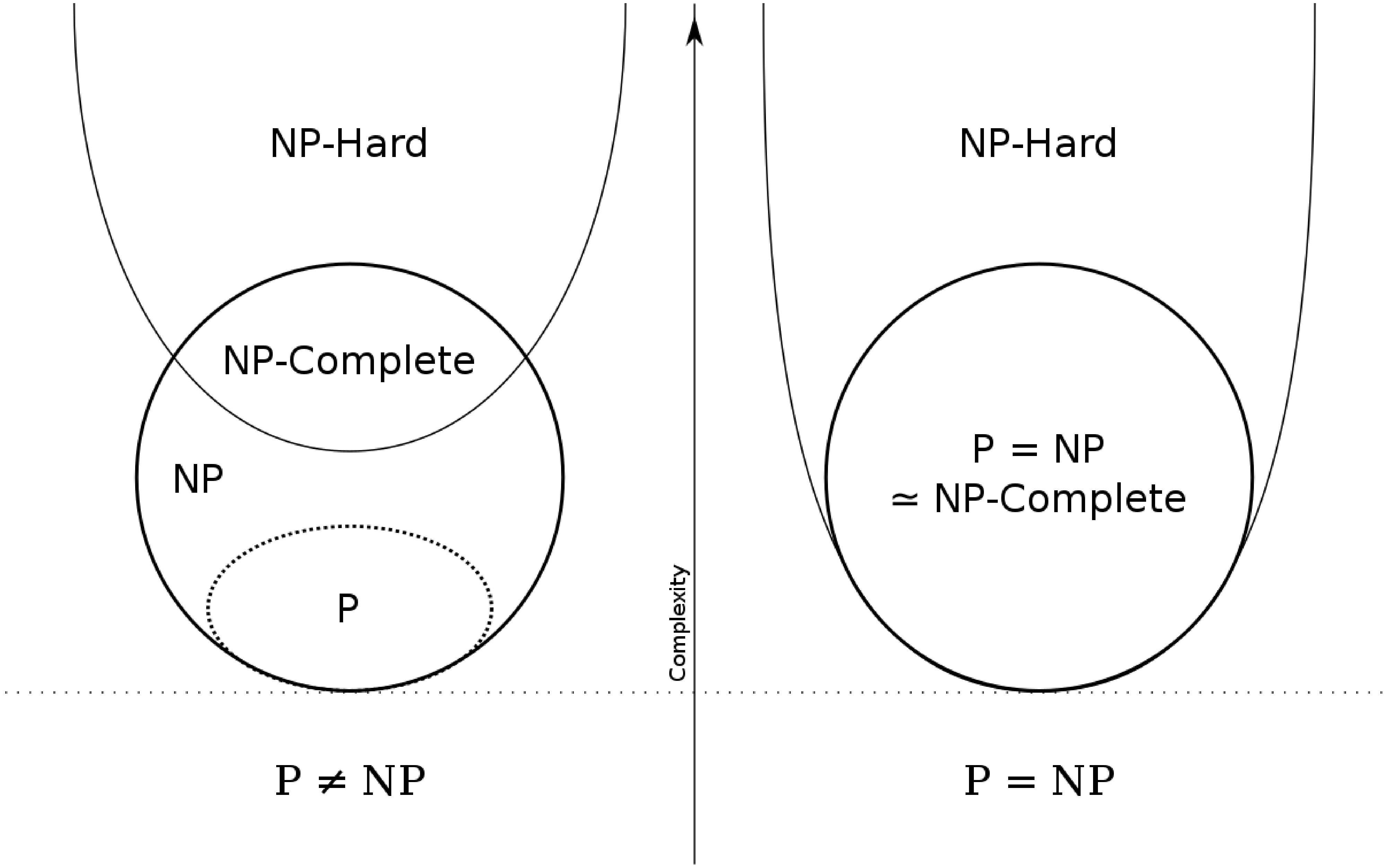
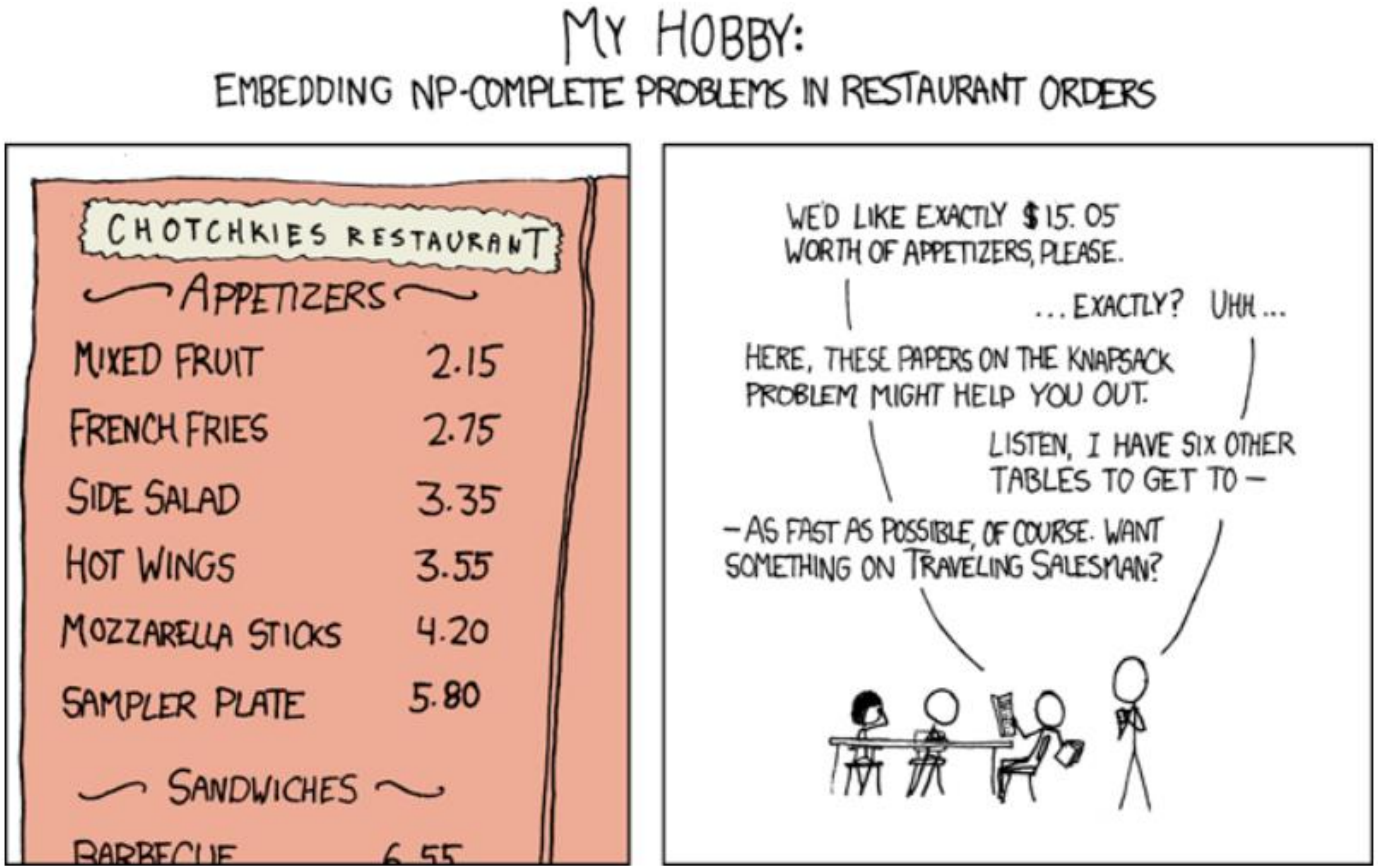


Image source: https://en.wikipedia.org/wiki/P_versus_NP_problem#/media/File:P_np_np-complete_np-hard.svg



Source: https://xkcd.com/287/