Teoria probabilităților - refresher

$A, B, C \dots$ - evenimente

$P(A)$ - probabilitatea ca evenimentul A să se întâmple

$$0 \leq P(A) \leq 1$$

$P(\bar{A}) = 1 - P(A)$ - probabilitatea inversă

A și B → notăm $P(A \cdot B)$

$$P(A \cdot B) = P(A|B) \cdot P(B)$$

Dacă A și B nu sunt legate cauzal:

$$P(A \cdot B) = P(A) \cdot P(B)$$

A sau B → $P(A + B)$

$$P(A+B) = P(A) + P(B) - P(A \cdot B)$$

Dacă A și B se exclud mutual:

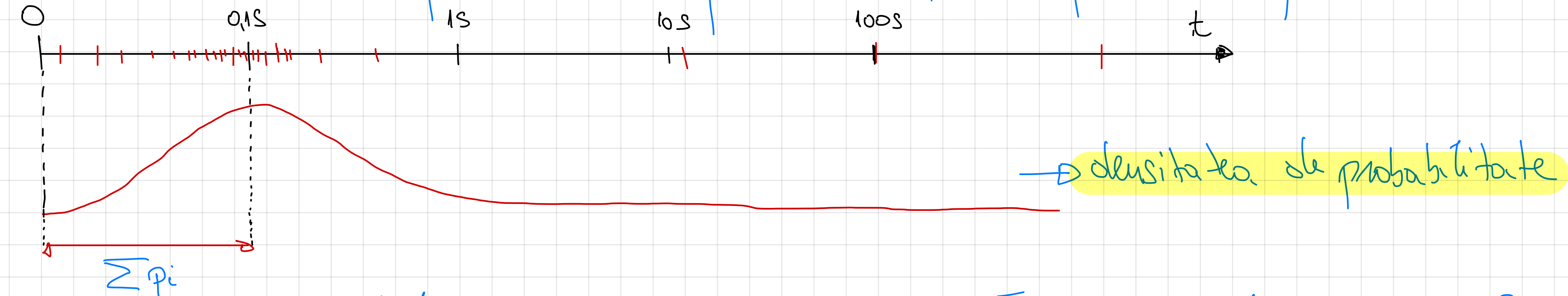
$$P(A+B) = P(A) + P(B)$$

Distribuții de probabilitate

Variabile aleatoare: X, Y, Z ... în spațiu de reprezentare < discret / continuu

Ex: timpul de răspuns al unui search engine → X

Dacă desenăm pe axa timpului toți timpii de răspuns:



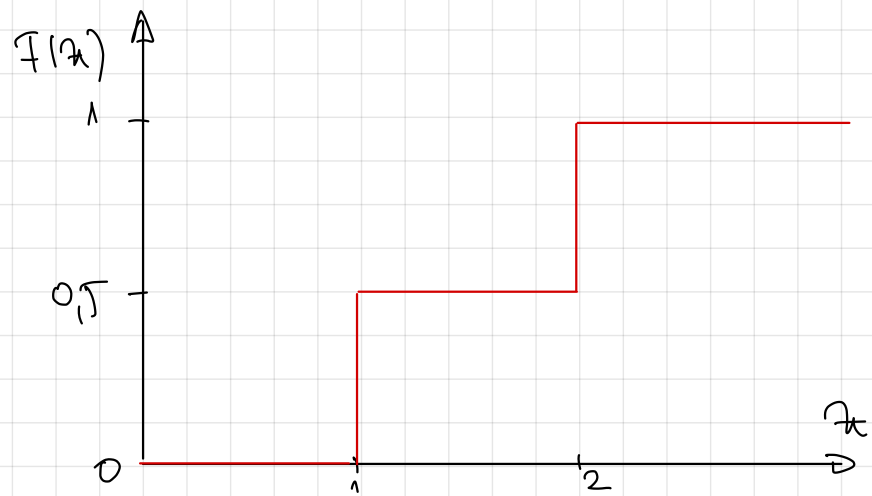
Care este probabilitatea ca timpul de răspuns să fie ≤ 0,1 s?

Sumă de probabilități → funcție cumulativă de distribuție a prob. (CDF)

CDF: $F_X(x) = P(X \le x)$

Ex: Aruncarea unei monede

$$F(x) = \begin{cases} 0, & x < 1 \\ 0{,}5, & x \in [1, 2) \\ 1, & x > 2 \end{cases}$$



Este relevant să folosim timpul ca variabilă aleatoare

$F(x) = F(t)$, $t \in [0, \infty)$

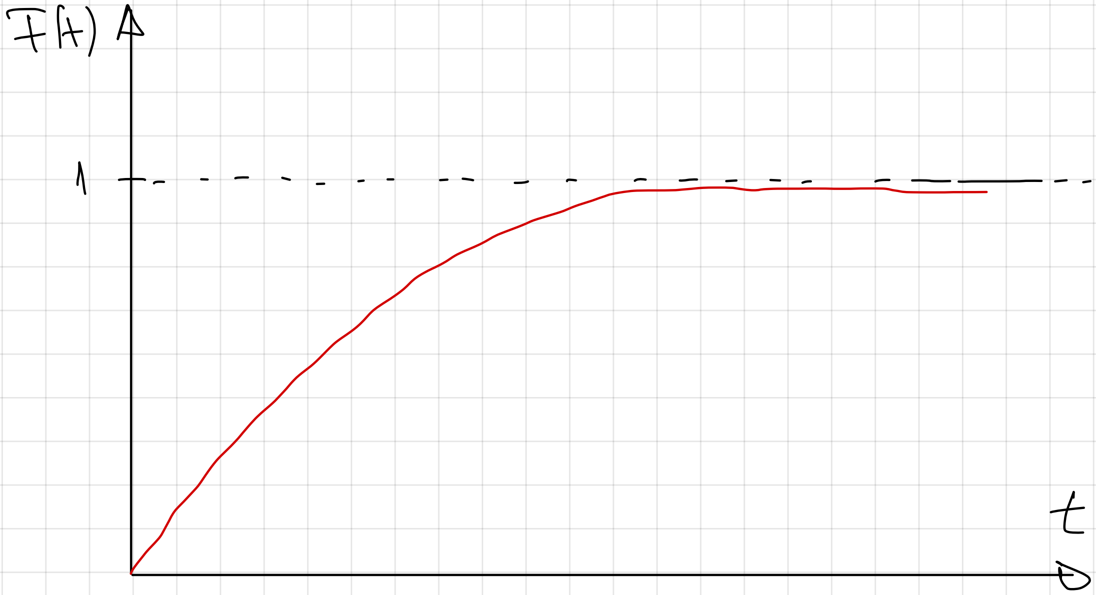
Proprietăți CDF:

$0 \le F(t) \le 1$

$F(0) = 0$

$\lim_{t \to \infty} F(t) = 1$

$F(t)$ este monoton crescătoare



Ex: $F(t) = 1 + 2e^{-3t} - 3e^{-2t}$

Funcția de densitate de probabilitate PDF

Notată $f(x)$ – probabilitatea ca un eveniment să se petreacă în $[t, t+\Delta t]$

Proprietate evidentă: $\int_0^\infty f(x)\,dx = 1$ și $f(x) = \frac{dF(x)}{dx}$

$$F(t) = \int_0^t f(\tau)\,d\tau$$

$$P(X \geq t) = \int_t^\infty f(\tau)\,d\tau$$

$$P(a \leq X \leq b) = \int_a^b f(\tau)\,d\tau = F(b) - F(a)$$

Ex: $F(t) = 1 + 2e^{-3t} - 3e^{-2t} \Rightarrow$

$f(t) = -6e^{-3t} + 6e^{-2t}$

Valoarea așteptată

$$E[X] = p_1 x_1 + p_2 x_2 + \dots p_n x_n = \frac{p_1 x_1 + \dots p_n x_n}{1} = \frac{p_1 x_1 + p_2 x_2 + \dots p_n x_n}{p_1 + p_2 + \dots p_n}$$

Ex: Pentru un zar cu 6 fețe:

$$E[X] = \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \dots + \frac{1}{6} \cdot 6 = 3,5$$

Dacă avem ca variabilă aleatoare timpul (var. continuă):

$$E[t] = \int_0^\infty t \cdot f(t)\, dt, \ \forall t \geq 0$$

Ex: $f(t) = -6e^{-3t} + 6e^{-2t} \Rightarrow E[t] = \int_0^\infty t \cdot \left(-6e^{-3t} + 6e^{-2t}\right) dt =$

$$= -6 \int_0^\infty t e^{-3t} dt + 6 \int_0^\infty t e^{-2t} dt = \dots = \frac{5}{6}$$

Distribuții de probabilitate

în funcție de tipul variabilei aleatoare — discrete
— continue

Distribuția binomială

Funcția de probabilitate a distribuției: $p(x)$

$$p(x) = C_n^x \, p^x (1-p)^{n-x}$$

Exemplu: Rată de defect la becuri 5%. Dacă inspectezi 100 de becuri, care e prob. să găsești 2 becuri defecte?

$$P(x=2) = C_{100}^{2} \, 0{,}05^2 (1-0{,}05)^{98} \simeq 0{,}081 \; (8{,}1\%)$$

Distribuția Poisson

$$P(x) = \frac{\lambda^x e^{-\lambda}}{x!}$$, unde λ este rata de apariție a unui eveniment

$\lambda = p \times n$

Exemplu: pentru ex. cu becurile o să avem $\lambda = 100 \cdot 0{,}05 = 5$

$$P(x=2) = \frac{5^2 e^{-5}}{2!} = \ldots = 0{,}09 \ (9\%)$$

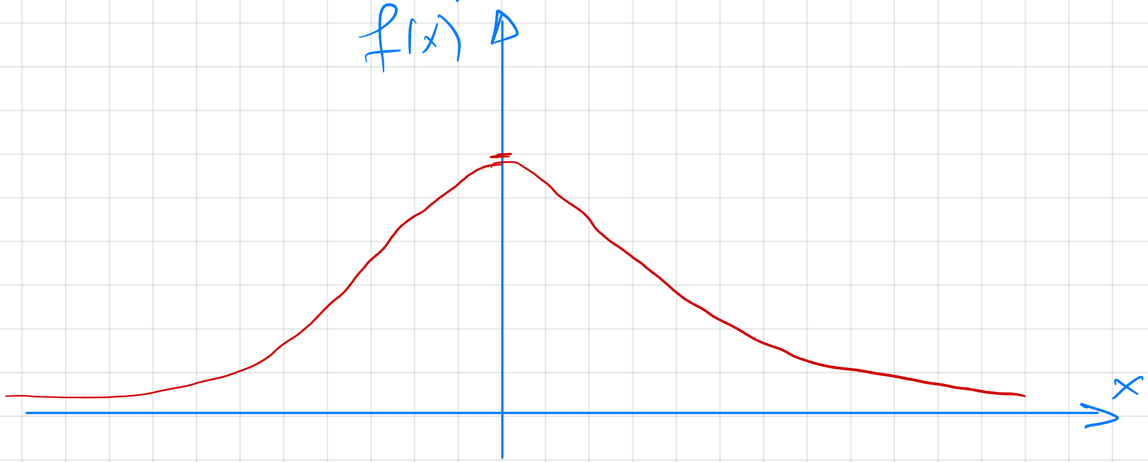
Distribuția normală (gaussiană)

Distribuție continuă → funcție densitate probabilitate

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)$$, unde σ – deviația standard

μ – medie

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n}(x_i-\mu)^2}{n-1}}$$



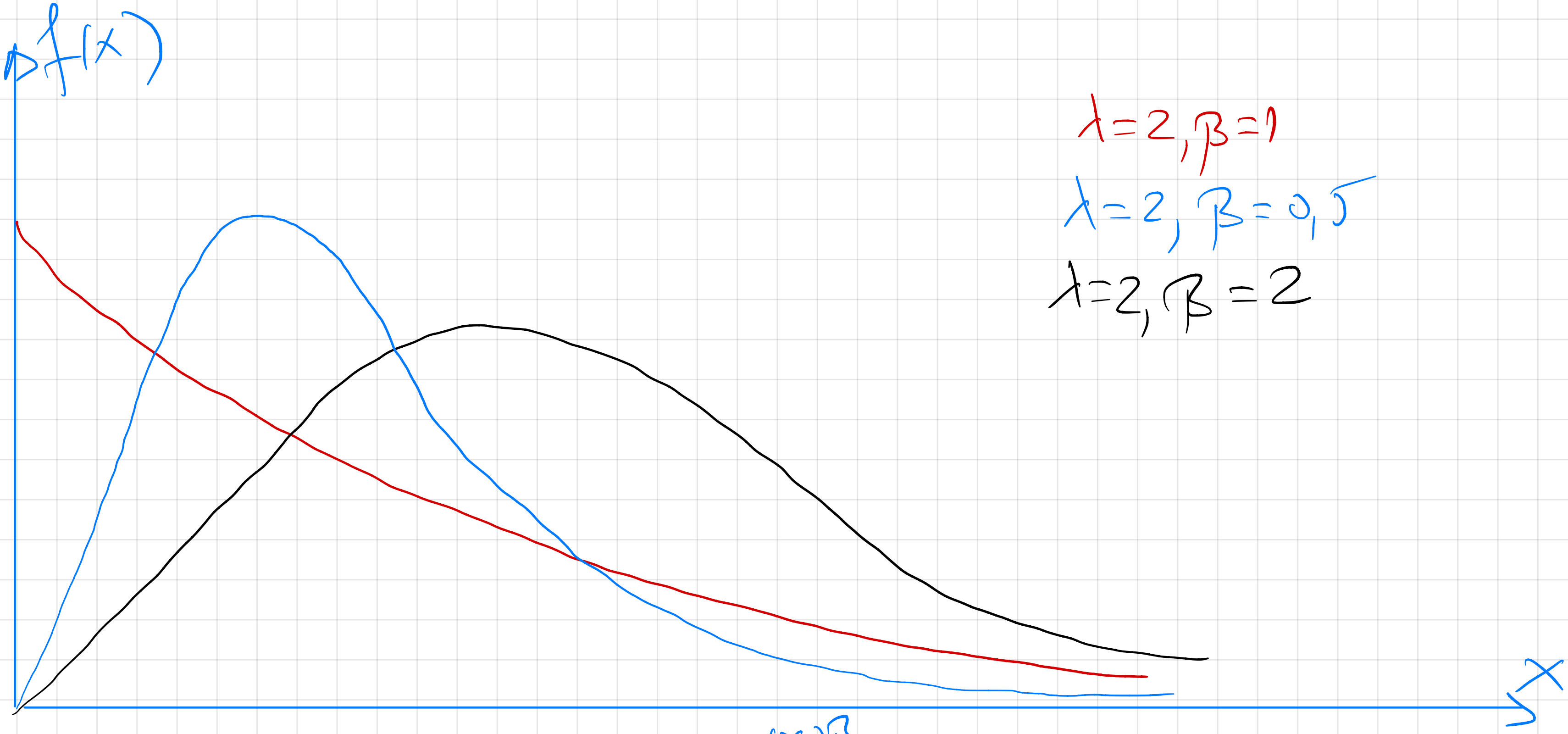
$$\mu = \frac{\sum_{i=1}^{n} x_i}{n}$$

Distributia Weibull

$$f(x) = \frac{\beta}{\lambda}\left(\frac{x}{\lambda}\right)^{\beta-1} e^{-\left(\frac{x}{\lambda}\right)^{\beta}}, \quad \text{unde}$$

$\lambda \rightarrow$ parametrul de scală

$\beta \rightarrow$ parametrul de formă

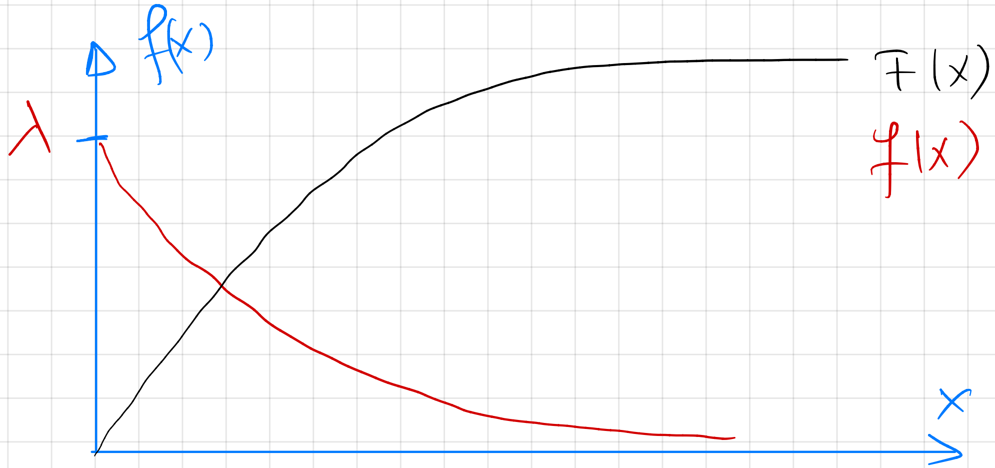


$$F(x) = \int f(x)\,dx = 1 - e^{-\left(\frac{x}{\lambda}\right)^{\beta}}$$

Distribuția exponențială

$$f(x) = \lambda e^{-\lambda x}$$ → Caz special al Weibull cu $\beta = 1$

$$F(x) = 1 - e^{-\lambda x}$$



Exemplu: server web, cererile vin aleatoriu. Avem o rată medie de 10 cereri pe minut ($\lambda = 10$).

Probabilitatea să primești o cerere în următorul minut:

$$F(1) = 1 - e^{-\lambda \cdot 1} = 1 - e^{-10} = 0,9995 \ (99,95\%)$$

$$f(1) = \lambda e^{-\lambda \cdot 1} = 10 e^{-10} \simeq 0,0005$$ → prob. cerere la minutul 1 (mică)

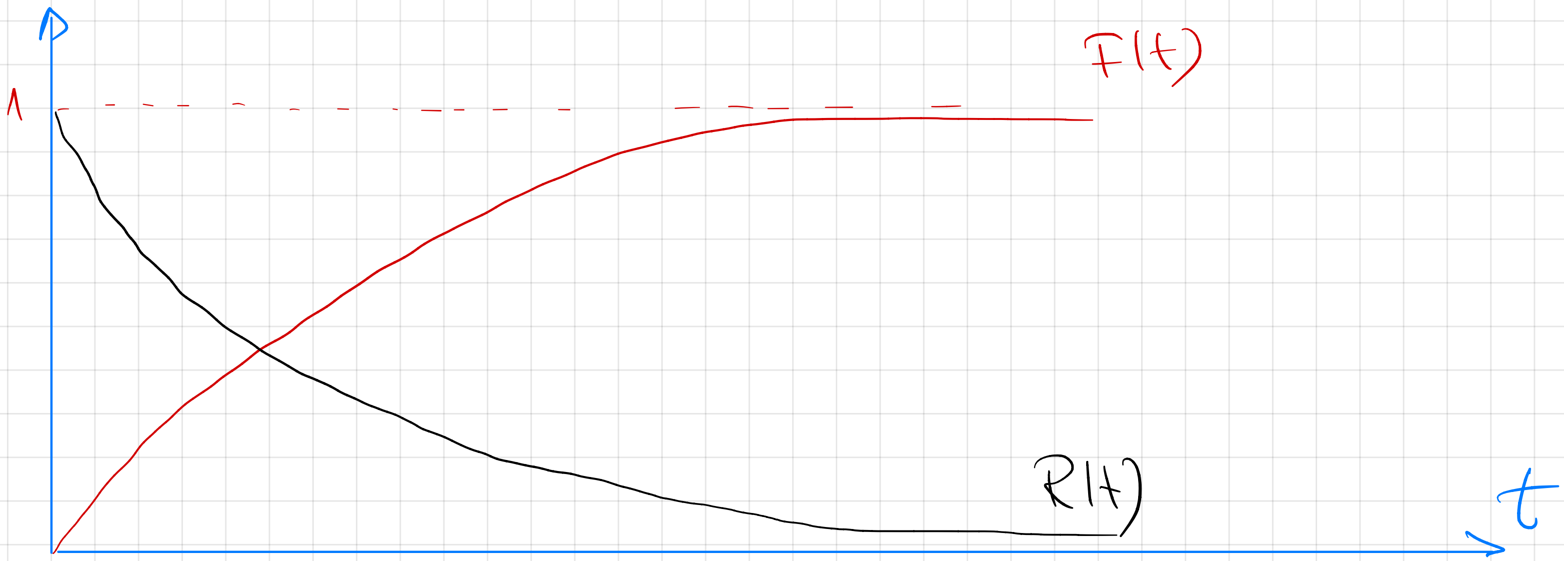
Modelarea fiabilității

Definim o funcție de fiabilitate $R(t)$

$R(t) = P($ sistemul este funcțional în $[0, t))$

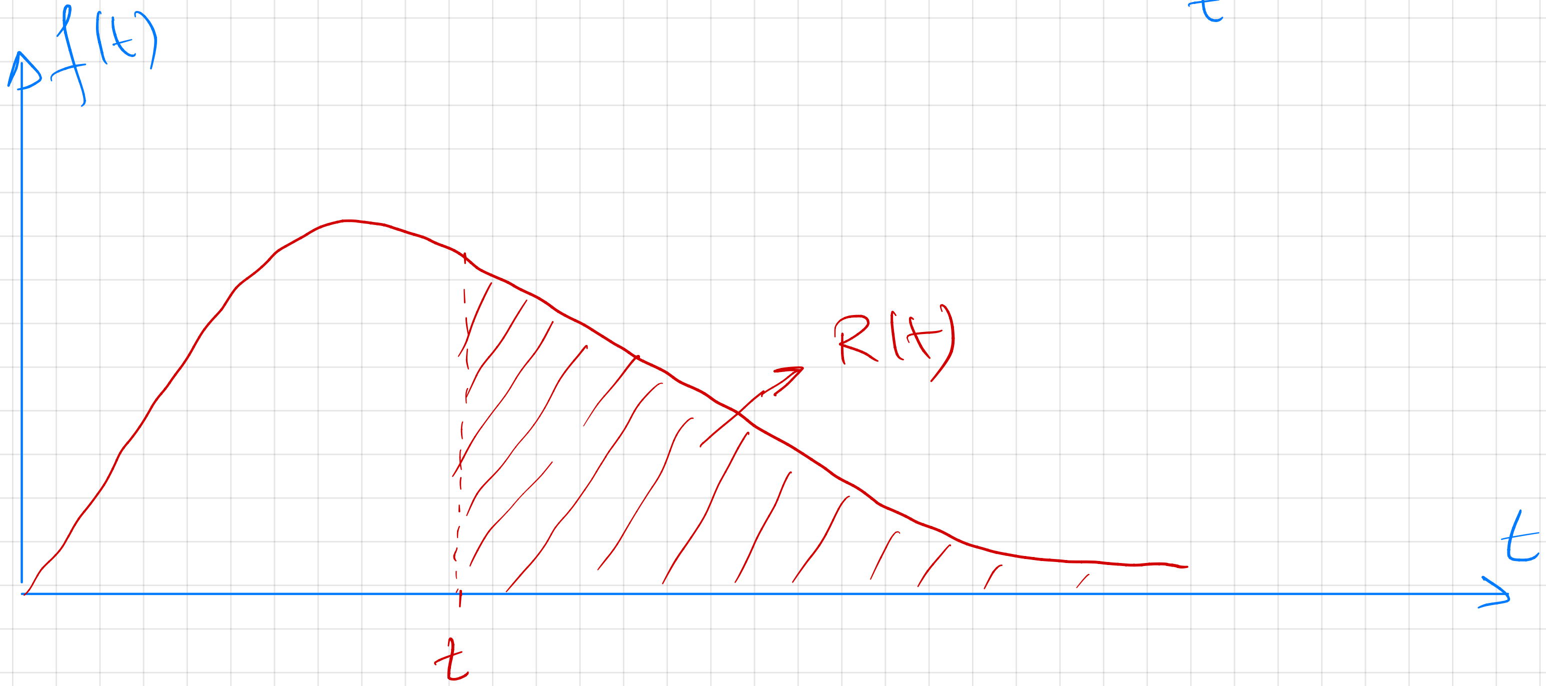
$F(t) \to$ CDF (prob. să apară un defect în $[0, t]$)

$$\left.\begin{aligned} R(t) &= P(X > t) \\ F(t) &= P(X \leq t) \end{aligned}\right| \Rightarrow R(t) = 1 - F(t)$$



Reprezentarea grafică a fiabilității

$$F(t) = \int_0^t f(z)\,dz \Rightarrow R(t) = \int_t^\infty f(z)\,dz$$



Funcția de fiabilitate poate fi interpretată ca aria de sub graficul funcției de densitate de probabilitate, de la un moment de timp la ∞.

Intensitatea defecțiunilor (failure rate)

Probabilitatea ca sistemul să se defecteze în $[t+\Delta t]$ cu condiția ca sistemul să funcționeze până la t.

$$P(t < X < t+\Delta t \mid X > t) = \frac{P(t < X < t+\Delta t)}{P(X > t)} = \frac{F(t+\Delta t) - F(t)}{1 - F(t)}$$

Definim intensitatea defecțiunilor:

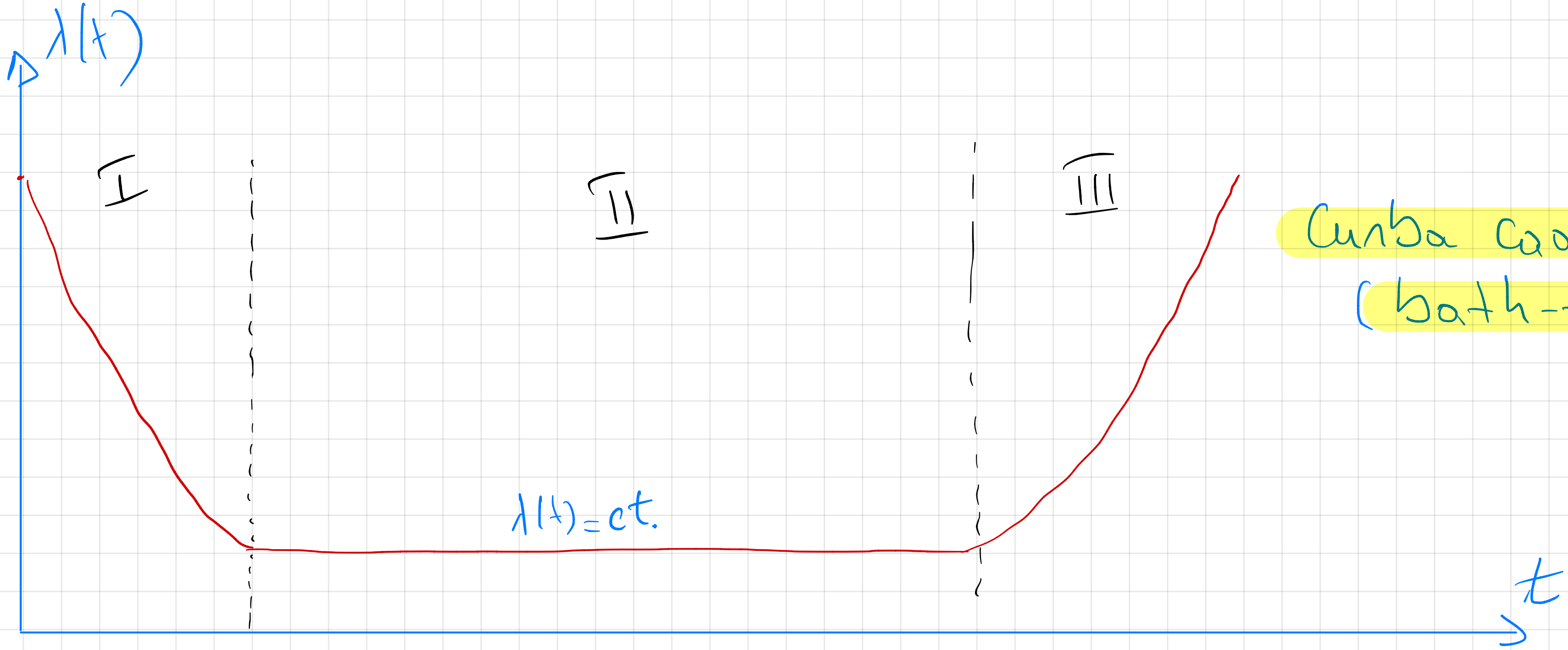
$$\lambda(t) = \lim_{\Delta t \to 0} \frac{F(t+\Delta t) - F(t)}{[1 - F(t)]\,\Delta t} = \lim_{\Delta t \to 0} \frac{R(t) - R(t+\Delta t)}{R(t)\,\Delta t}$$

dar $R(t) = 1 - F(t) \Rightarrow$

$$\lambda(t) = \frac{1}{R(t)} \lim_{\Delta t \to 0} \frac{R(t) - R(t+\Delta t)}{\Delta t} = \frac{1}{R(t)} \left[-\frac{dR(t)}{dt} \right] = \frac{f(t)}{R(t)}$$

Dar $f(t) = \dfrac{dF(t)}{dt} = \dfrac{d(1 - R(t))}{dt} = -\dfrac{dR(t)}{dt}$

Deci $\boxed{\lambda(t) = \frac{f(t)}{R(t)}}$, sau $\boxed{\lambda(t) = -\frac{1}{R(t)} \frac{d(R(t))}{dt}}$



Curba cadă de baie
(bath-tub curve)

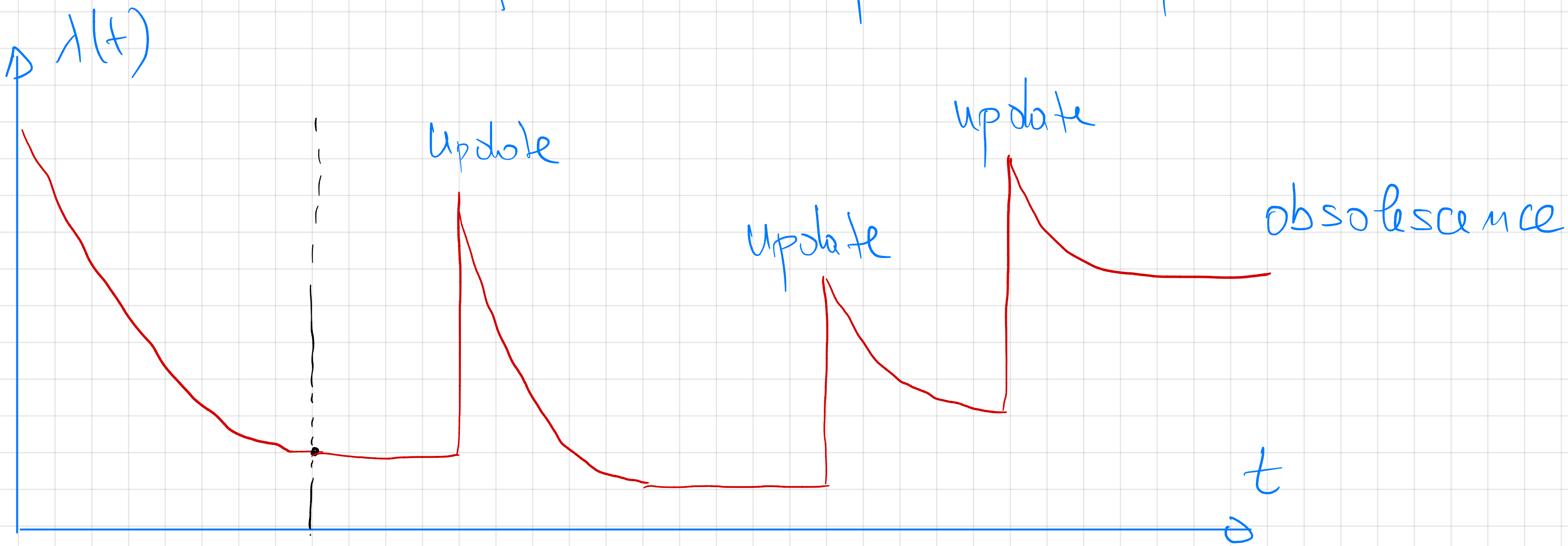
Zone distincte în profilul $\lambda(t)$:

I. Mortalitate infantilă

II. Zona de utilizare normală a produsului (λ=ct)

III. Îmbătrânire (wear-out)

Intensitatea defecțiunilor pentru software



Dacă $\lambda(t) = ct. = \lambda \Rightarrow \lambda(t) = \frac{f(t)}{R(t)} = -\frac{1}{R(t)} \frac{dR(t)}{dt} \Rightarrow$

$$\Rightarrow \lambda = -\frac{1}{R(t)} \frac{dR(t)}{dt} \Rightarrow -\lambda\, dt = \frac{1}{R(t)} dR(t) \;\Big|\cdot\int \Rightarrow -\lambda \int dt = \int \frac{1}{R(t)} dR(t)$$

$$\Rightarrow -\lambda(t+c) = \ln(R(t)) \Rightarrow R(t) = e^{-\lambda(t+c)}$$

Dar $R(0) = 1 \Rightarrow e^{-\lambda(0+c)} = 1 \Rightarrow e^{-\lambda \cdot c} = 1 \Big|_{\lambda \neq 0} \Rightarrow c = 0 \Rightarrow \boxed{R(t) = e^{-\lambda t}}$

Dacă $\lambda(t)$ nu este constant, modelăm folosind distribuția Weibull

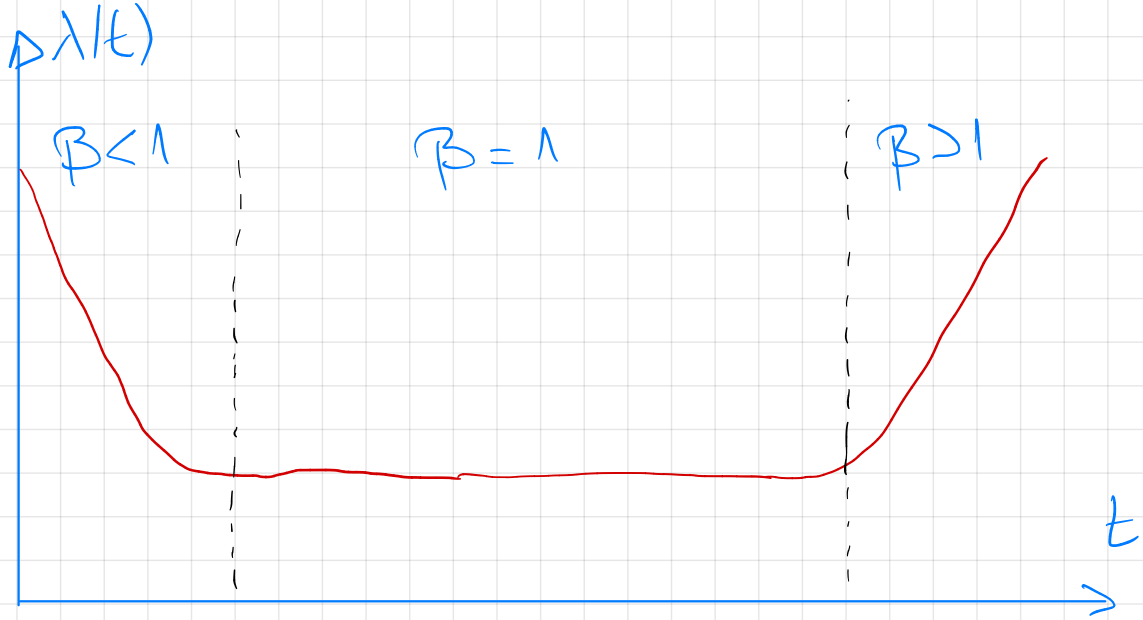
$$\lambda(t) = \lambda\beta t^{\beta-1}$$

$\beta > 1$: $\lambda(t) \nearrow$ – îmbătrânire

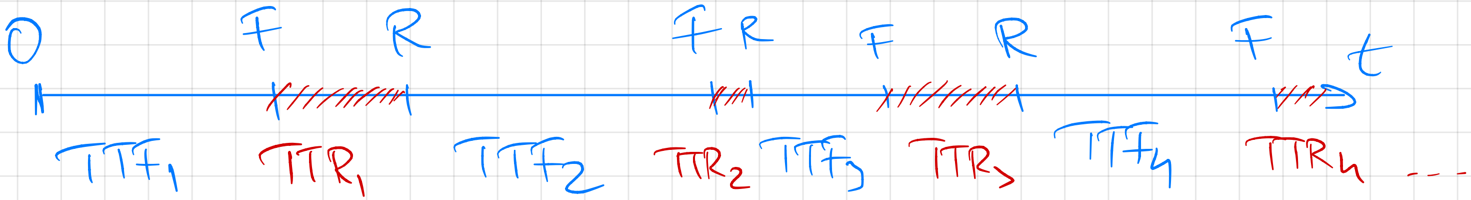
$\beta = 1$: $\lambda(t) = ct = \lambda$

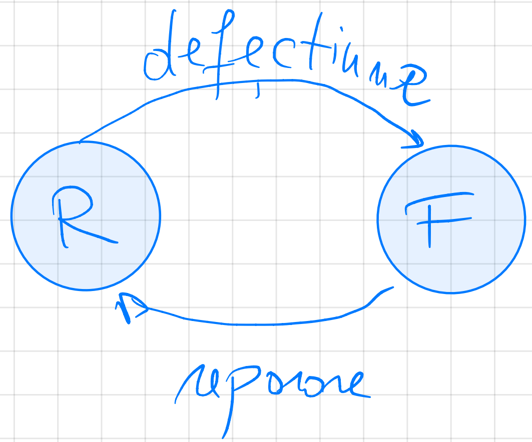
$\beta < 1$: $\lambda(t) \downarrow$ – mortalitate infantilă

$$R(t) = e^{-\lambda t^{\beta}}$$



Media timpului de bună funcționare





$$MTBF = \frac{\sum TTF_i}{n} \qquad MTTR = \frac{\sum TTR_i}{n}$$

A - disponibilitate

$$A = \frac{\sum TTF_i}{\sum TTF_i + \sum TTR_i} = \frac{MTBF}{MTBF + MTTR}$$

Exemplu: MTBF = 2 ore (7200 s), MTTR = 3 s

$$A = \frac{7200}{7200+3} = 99,96\%$$

Exemplu:

Server email: 99,99% disponibilitate 4 de 9 → down 52,56 min/an

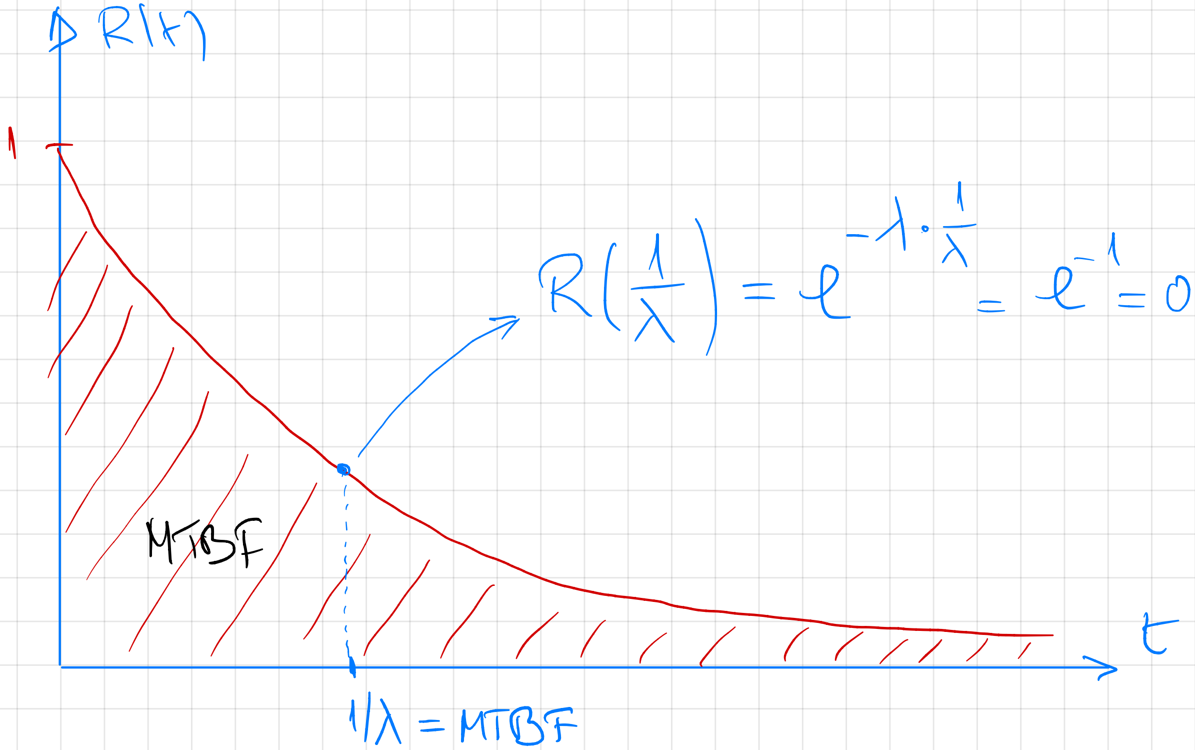
Sisteme critice: 99,9999%, 6 de 9, va fi indisponibil 31,5 s/an

$$MTBF = E[T] = \int_0^\infty t \cdot f(t)\, dt$$

Dan $f(t) = -\dfrac{dR(t)}{dt} \Rightarrow MTBF = \displaystyle\int_0^\infty t \cdot \left(-\dfrac{dR(t)}{dt}\right) dt =$

$$= -\int_0^\infty t \cdot R'(t)\, dt = -\Big[\underbrace{t \cdot R(t)\Big|_0^\infty}_{=0} - \int_0^\infty R(t)\, dt \Big] = \int_0^\infty R(t)\, dt$$

$$MTBF = \int_0^\infty R(t)\, dt$$



$$R\left(\frac{1}{\lambda}\right) = e^{-\lambda \cdot \frac{1}{\lambda}} = e^{-1} = 0{,}37$$

Dacă $R(t) = e^{-\lambda t} \Rightarrow MTBF = \int_0^\infty e^{-\lambda t} dt = -\frac{1}{\lambda} e^{-\lambda t} \Big|_0^\infty =$

$= -\frac{1}{\lambda}(0 - 1) = \frac{1}{\lambda}$

$$MTBF = \frac{1}{\lambda}$$

Dacă $\lambda(t) \neq ct \Rightarrow R(t) = e^{-\lambda t^\beta}$ (distribuția Weibull)

$MTBF = \int_0^\infty R(t)\, dt = \int_0^\infty e^{-\lambda t^\beta} dt = \frac{\Gamma(\beta^{-1})}{\beta \lambda^{\beta - 1}}$

$\Gamma(x)$ - funcția gamma - generalizarea factorialei pt $x \in \mathbb{R}$

$\Gamma(0) = \Gamma(1) = 1$

$\Gamma(x+1) = x \cdot \Gamma(x)$, $\forall x > 1$

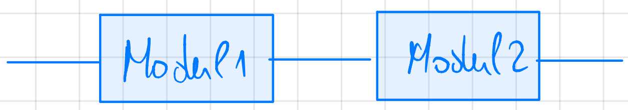
Dacă x este întreg pozitiv $\Gamma(x) = (x-1)!$

$\Gamma(x) = \int_0^\infty y^{x-1} e^{-y} dy$

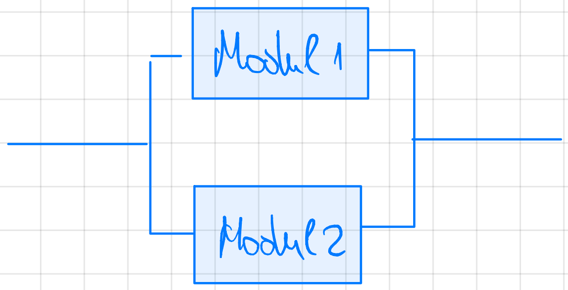
Modelarea fiabilității – diagrame bloc

Folosim diagrame bloc pentru a descrie un sistem din punct de vedere fiabilistic

De exemplu:



structuri serie



Structuri paralel

$R_i(t)$ – fiabilitatea modulului i

$Q_i(t)$ – inversul fiabilității, $Q_i(t) = 1 - R_i(t)$

Structura Serie

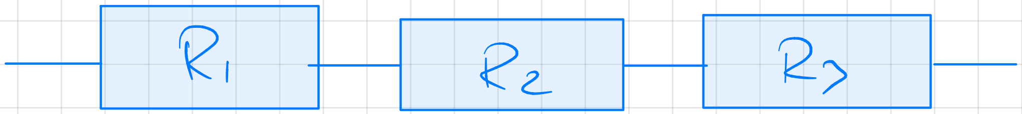
$$R_{serie} = R_1 \cdot R_2 \cdot \ldots \cdot R_n = \prod_{i=1}^{n} R_i$$

$$Q_{serie} = 1 - R_{serie} = 1 - \prod_{i=1}^{n} (1 - Q_i) = 1 - \Big(1 - (Q_1 + Q_2 + \ldots Q_n) + (Q_1Q_2 + \ldots Q_{n-1}Q_n) -$$

$$- (Q_1Q_2Q_3 + \ldots) + \ldots \prod_{i=1}^{n} Q_i\Big) \simeq 1 - \Big(1 - (Q_1 + Q_2 + \ldots Q_n)\Big) = \sum_{i=1}^{n} Q_i$$

$$Q_{serie} \simeq \sum_{i=1}^{n} Q_i$$

Exemplu: 3 module cu fiabilitate R_1, R_2 şi R_3



La un anumit timp t: $R_1 = 0,9$, $R_2 = 0,3$; $R_3 = 0,5$

$$R_{serie} = R_1 \cdot R_2 \cdot R_3 = 0,9 \cdot 0,3 \cdot 0,5 = 0,27 \cdot 0,5 = 0,13$$

Dacă $R_1 = R_2 = R_3 = 0,99 \Rightarrow R_{serie} = 0,99^3 \simeq 0,97$ (97%)

$$Q_{serie} \simeq Q_1 + Q_2 + Q_3 = 0,01 + 0,01 + 0,01 = 0,03$$

Dacă $R_i(t) = e^{-\lambda_i t}$

$$R_{serie} = \prod_{i=1}^{n} R_i(t) = \prod_{i=1}^{n} e^{-\lambda_i t} = e^{-\sum_{i=1}^{n} \lambda_i t} = e^{-\lambda_s t}$$

$$\boxed{\lambda_{serie} = \sum_{i=1}^{n} \lambda_i}$$

$$MTBF_{serie} = \int_0^{\infty} R_{serie}(t)\,dt = \int_0^{\infty} e^{-\sum_{i=1}^{n} \lambda_i t}\,dt = \frac{1}{\sum_{i=1}^{n} \lambda_i} = \frac{1}{\lambda_s}$$

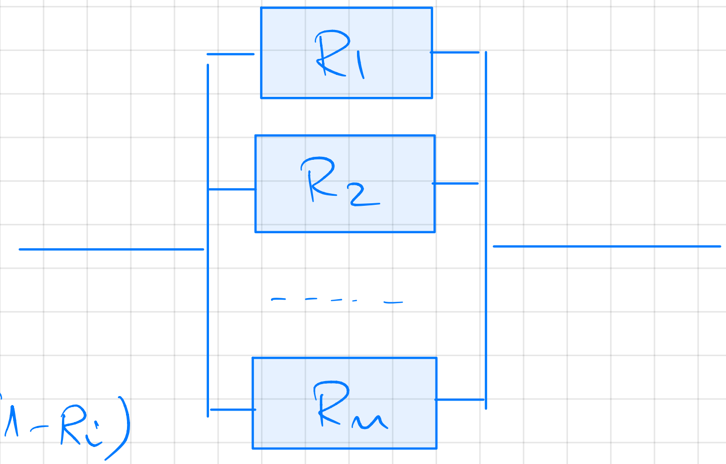
$$\boxed{MTBF_{serie} = \frac{1}{\lambda_s}}$$

Exemplu: 3 module cu $MTBF_1 = 5h$, $MTBF_2 = 7h$ și $MTBF_3 = 2h$

$$MTBF_{serie} = \frac{1}{\lambda_1 + \lambda_2 + \lambda_3} = \frac{1}{\frac{1}{MTBF_1} + \frac{1}{MTBF_2} + \frac{1}{MTBF_3}} = \frac{1}{{}^{14/}\frac{1}{5} + {}^{10/}\frac{1}{7} + {}^{35/}\frac{1}{2}} = \frac{1}{\frac{14+10+35}{70}} =$$

$$= \frac{70}{59} = 1{,}186\,h$$

Structura paralel



$$R_{paralel} = ?$$

$$Q_{paralel} = Q_1 \cdot Q_2 \cdot \ldots \cdot Q_n = \prod_{i=1}^{n} Q_i$$

$$R_{paralel} = 1 - Q_{paralel} = 1 - \prod_{i=1}^{n} Q_i = 1 - \prod_{i=1}^{n} (1 - R_i)$$

Exemplu

: $R_1 = 0,9$; $R_2 = 0,7$; $R_3 = 0,5$

$$R_{paralel} = 1 - (1-R_1)(1-R_2)(1-R_3) = 1 - 0,1 \cdot 0,7 \cdot 0,5 = 1 - 0,035 = 0,965$$

96,5 % !

Dacă $R_1 = R_2 = R_3 = 0,99$

$$R_{paralel} = 1 - (1 - 0,99)^3 = 1 - 0,01^3 = 1 - 0,000001 = 99,99\ldots\ \%$$

Dacă $R_i = e^{-\lambda_i t} \Rightarrow$

$$R_{paralel} = 1 - \prod_{i=1}^{n} \left(1 - e^{-\lambda_i t}\right)$$

$$MTBF_{paralel} = \int_0^{\infty} R_{paralel}(t)\,dt = \int_0^{\infty} \left[1 - \prod_{i=1}^{n} \left(1 - e^{-\lambda_i t}\right)\right] dt =$$

$$= \int_0^\infty \sum_{i=1}^{n} e^{-\lambda_i t} dt - \int_0^\infty \sum_{\substack{i,j=1\\ i\neq j}}^{n} e^{-(\lambda_i+\lambda_j)t} + \int_0^\infty \dots + (-1)^{n+1} \int_0^\infty \prod_{i=1}^{n} e^{-\lambda_i t} =$$

$$= \sum_{i=1}^{n} \frac{1}{\lambda_i} - \sum_{\substack{i,j=1\\ i\neq j}}^{n} \frac{1}{\lambda_i+\lambda_j} + \dots + (-1)^{n+1} \frac{1}{\sum_{i=1}^{n} \lambda_i}$$

Dacă sistemele sunt identice $\Rightarrow R_i(t) = e^{-\lambda t}$, $MTBF_i = MTBF = \frac{1}{\lambda}$

$$MTBF_{paralel} = \frac{n}{\lambda} - \frac{n}{2\lambda} + \dots (-1)^{n+1} \frac{1}{n\lambda} = \frac{1}{\lambda}\left(n - \frac{n}{2} + \frac{n}{3} - \dots + (-1)^{n+1}\frac{1}{n}\right) =$$

$$= \frac{1}{\lambda} \cdot \underbrace{\sum_{i=1}^{n} (-1)^{i+1} \frac{C_n^i}{i}}_{= \sum_{i=1}^{n} \frac{1}{i}} = \frac{1}{\lambda} \sum_{i=1}^{n} \frac{1}{i} \simeq \frac{\ln 2n}{\lambda}$$

$$\boxed{MTBF_{paralel} = MTBF \cdot \sum_{i=1}^{n} \frac{1}{i}}$$

Exemplu: $MTBF_1 = 5h$, $MTBF_2 = 7h$ și $MTBF_3 = 2h$

$\lambda_1 = 1/MTBF_1 = 1/5$, $\lambda_2 = 1/MTBF_2 = 1/7$ și $\lambda_3 = 1/MTBF_3 = 1/2$

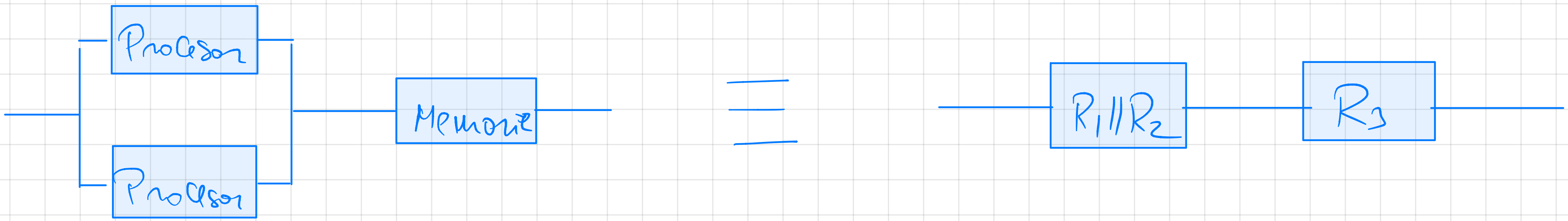
$$MTBF_{paralel} = \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3}\right) - \left(\frac{1}{\lambda_1+\lambda_2} + \frac{1}{\lambda_1+\lambda_3} + \frac{1}{\lambda_2+\lambda_3}\right) + \frac{1}{\lambda_1+\lambda_2+\lambda_3} =$$

$$= (5+7+2) - \left(\frac{1}{\frac{1}{5}+\frac{1}{7}} + \frac{1}{\frac{1}{5}+\frac{1}{2}} + \frac{1}{\frac{1}{7}+\frac{1}{2}}\right) + \frac{1}{\frac{1}{5}+\frac{1}{7}+\frac{1}{2}} =$$

$$= 14 - \left(\frac{35}{12} + \frac{10}{7} + \frac{14}{9}\right) + \frac{70}{59} = 14 - (2{,}92 + 1{,}43 + 1{,}56) + 1{,}19 = 9{,}28\ h$$

Combinații serie – paralel

Exemplu: procesor dual-core cu o singură memorie RAM



R_1, R_2, R_3 $\Rightarrow$ $R_{TOTAL} = (R_1 \| R_2) \cdot R_3 = \left(1 - (1-R_1)(1-R_2)\right) \cdot R_3$

$= (R_1 + R_2 - R_1 R_2) \cdot R_3 = R_1 R_3 + R_2 R_3 - R_1 R_2 R_3$

$R_1 = 0{,}9$

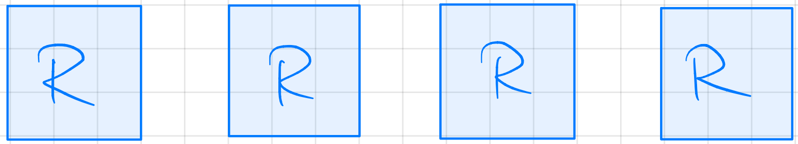
$R_2 = 0{,}3$ $\Rightarrow$ $R_{TOTAL} = \ldots$

$R_3 = 0{,}7$

Structuri k din n

Exemplu: avion pasageri cu 4 motoare, tolerează maxim 2 motoare defecte.

$$R_{2/4} = R^4 + 4R^3(1-R) + 6R^2(1-R)^2$$



Caz general:

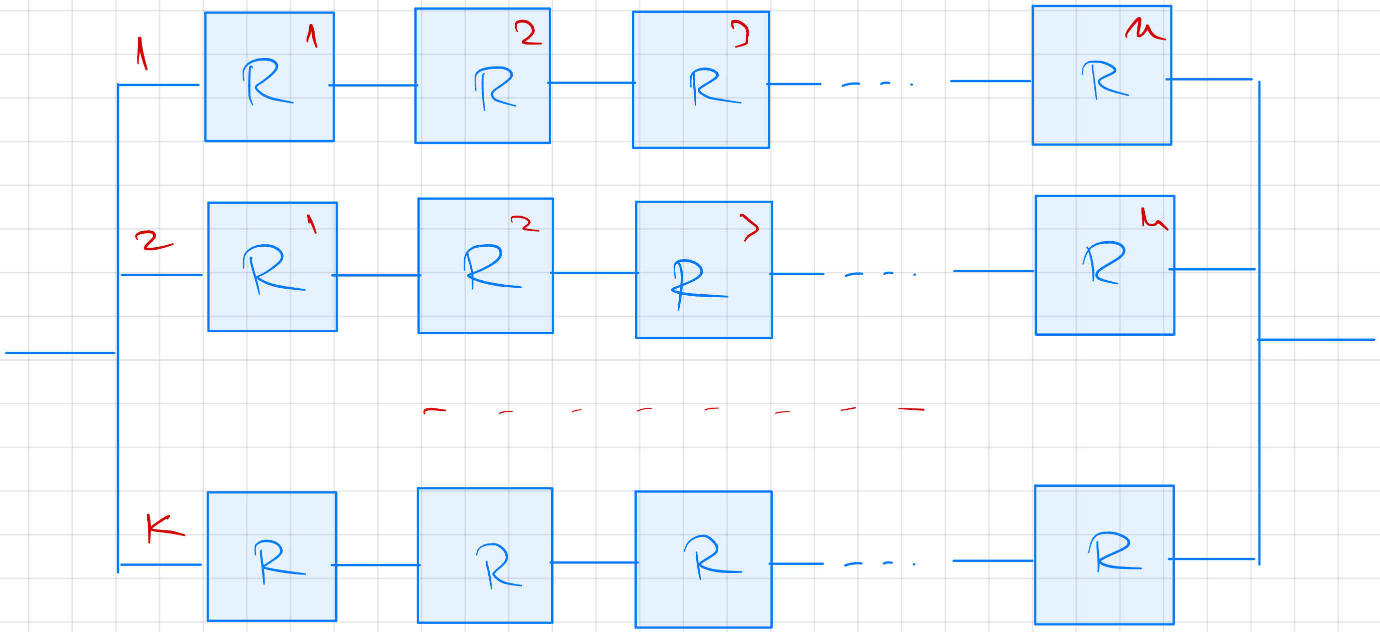
$$R_{k/n} = \sum_{i=k}^{n} C_n^i R^i (1-R)^{n-i}$$

$$R_{1/n} = \sum_{i=1}^{n} C_n^i R^i (1-R)^{n-i}$$ – structura paralel

$$R_{n/n} = R^n$$ – structura serie

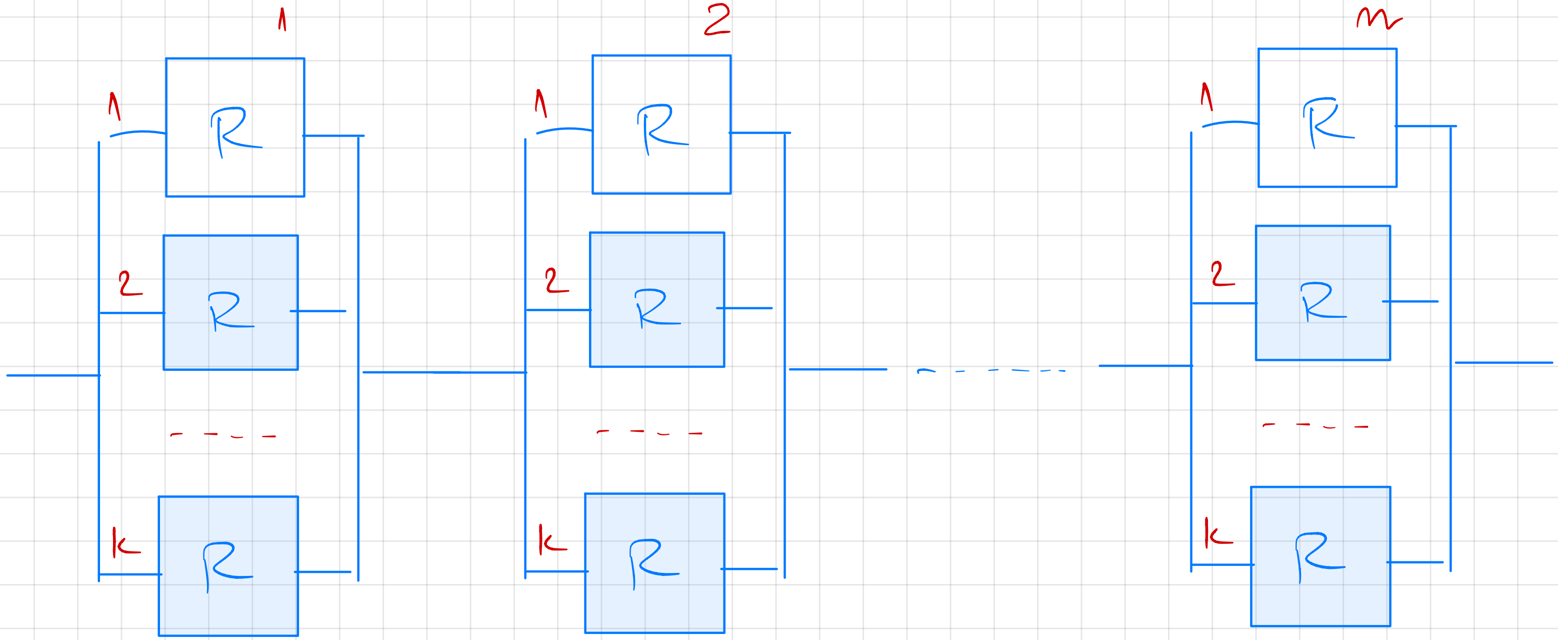
Structuri serie-paralel și paralel-serie

Serie-paralel



$$R_{SP}(t) = \underbrace{R^n // R^n // \ldots // R^n}_{K \text{ ori}} = 1 - \left(1 - R^n(t)\right)^K$$

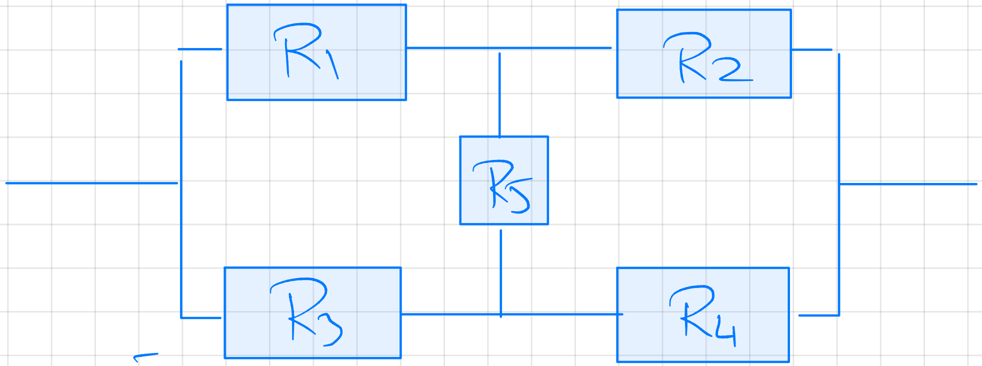
Parallel-Serie



$$R_{PS}(t) = \left[1-(1-R(t))^{k}\right]^{m}$$

Structuri mixte compozabile

$R_{TOTAL} = ?$

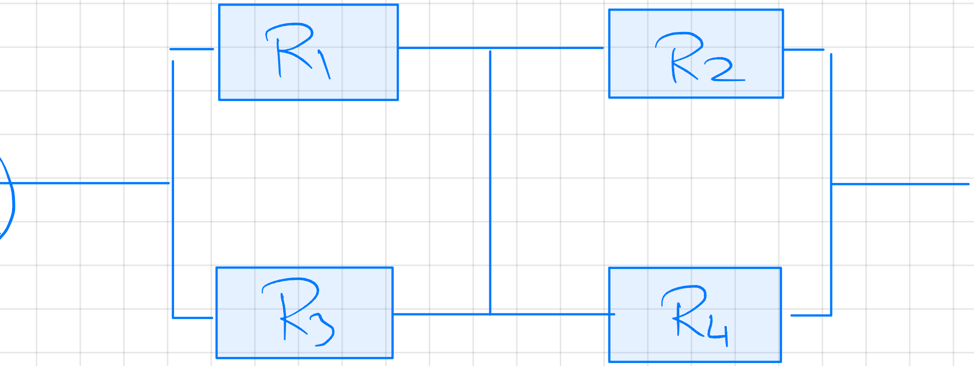


Cazul 1: Modulul 5 funcționează

$R_5(t) = 1$

$$R_{CAZ1} = (R_1 \| R_3) \cdot (R_2 \| R_4) =$$

$$= (R_1 + R_3 - R_1R_3)(R_2 + R_4 - R_2R_4)$$

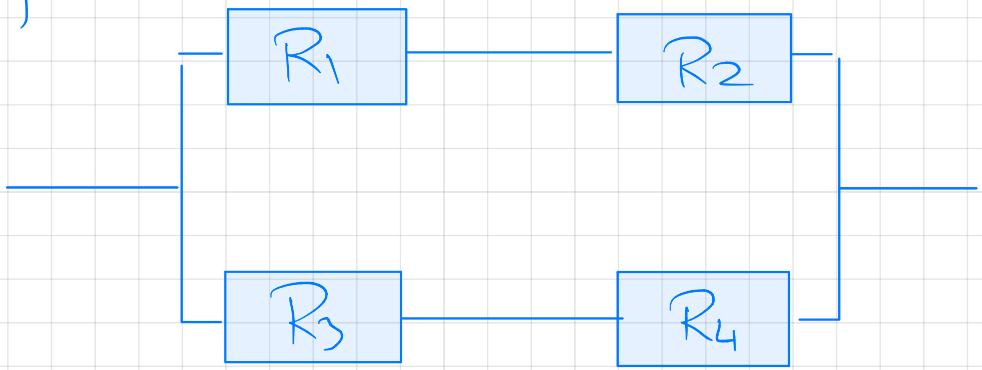


Cazul 2: modulul 5 este defect

$R_5(t) = 0$

$$R_{CAZ2} = (R_1 \cdot R_2) \| (R_3 \cdot R_4) =$$

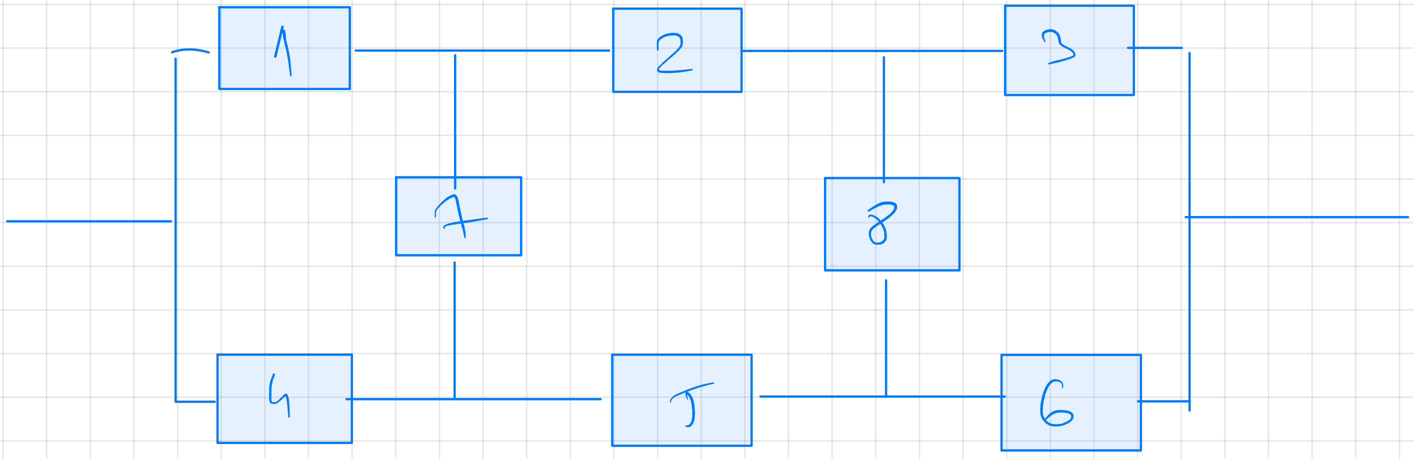
$$= R_1R_2 + R_3R_4 - R_1R_2R_3R_4$$



$$R_{TOTAL} = R_5 \cdot R_{CAZ1} + (1-R_5) R_{CAZ2} =$$

$$= R_5 \left[(R_1 + R_3 - R_1 R_3)(R_2 + R_4 - R_2 R_4) \right] + (1-R_5)(R_1 R_2 + R_3 R_4 - R_1 R_2 R_3 R_4)$$

Alt exemplu:



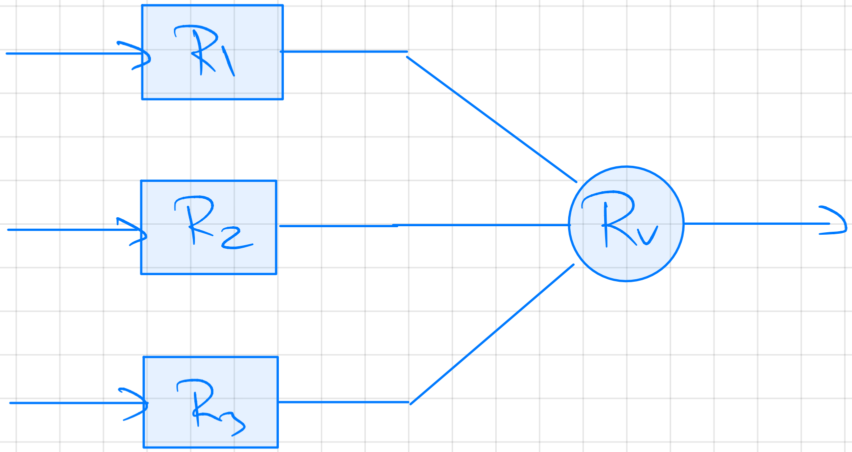
Stare	7	8
S1	OK	OK
S2	OK	FAIL
S3	FAIL	OK
S4	FAIL	FAIL

$$R_{TOTAL} = R_7 \cdot R_8 \cdot R_{S_1} + R_7 (1-R_8) R_{S_2} +$$

$$+ (1-R_7) R_8 R_{S_3} + (1-R_7)(1-R_8) R_{S_4}$$

Structuri cu votare majoritară

Structuri cu votare 2/3



De obicei voter-ul este mult mai simplu decât unitatea de calcul

$$R_V >> R_1, R_2, R_3 \Rightarrow R_V \simeq 1$$

$$R_{2/3} = \left[R_1R_2R_3 + (1-R_1)R_2R_3 + R_1(1-R_2)R_3 + R_1R_2(1-R_3)\right] \cdot R_V$$

Dacă $R_1 = R_2 = R_3 = R \Rightarrow$

$$R_{2/3} = R^3 + 3(1-R)R^2 = R^3 + 3R^2 - 3R^3 = 3R^2 - 2R^3$$

$$R_{2/3} = 3R^2 - 2R^3$$

Dacă $R = 0{,}9 \Rightarrow R_{2/3} = 3 \cdot 0{,}9^2 - 2 \cdot 0{,}9^3 = 3 \cdot 0{,}81 - 2 \cdot 0{,}729 = 0{,}972$ (97,2%)

Dacă $R = 0{,}1 \Rightarrow R_{2/3} = 3 \cdot 0{,}1^2 - 2 \cdot 0{,}1^3 = 3 \cdot 0{,}01 - 2 \cdot 0{,}001 = 0{,}03 - 0{,}002 = 0{,}028$ (2,8%)

Când $R_{2/3} > R$?

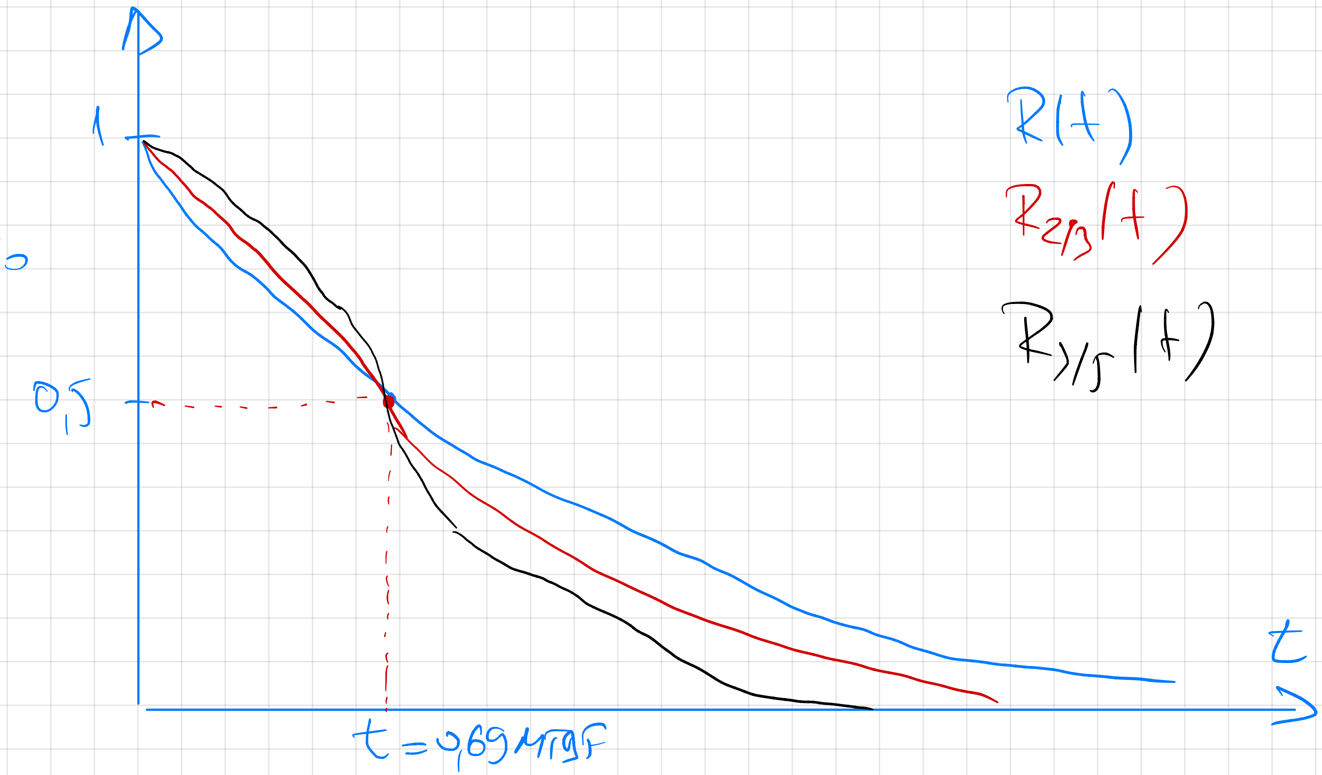
$$3R^2 - 2R^3 > R \Rightarrow 2R^3 - 3R^2 + R < 0 \Rightarrow R(2R^2 - 3R + 1) < 0$$

Dar $R \in [0,1] \Rightarrow 2R^2 - 3R + 1 < 0 \Rightarrow (R-1)\left(R-\frac{1}{2}\right) < 0$

R		0		$\frac{1}{2}$		1	
$R-1$	$-$	$-$	$-$	$-$	$-$	0	$+$
$R-\frac{1}{2}$	$-$	$-$	$-$	0	$+$	$+$	$+$
P	$+$	$+$	$+$	0	$-$	0	$+$

$R \in (1/2,\ 1)$

$R_{2/3} > R\ (\forall)\ R > 50\%$



Dokaz $R(t) = e^{-\lambda t}$

$R_{2/3}(t) = 3 \cdot e^{-2\lambda t} - 2e^{-3\lambda t}$

$R(t) > 0{,}5 \Rightarrow e^{-\lambda t} > 0{,}5 \Rightarrow e^{-\lambda t} = \frac{1}{2} \Rightarrow -\lambda t = \ln\frac{1}{2} \Rightarrow \lambda t = \ln 2$

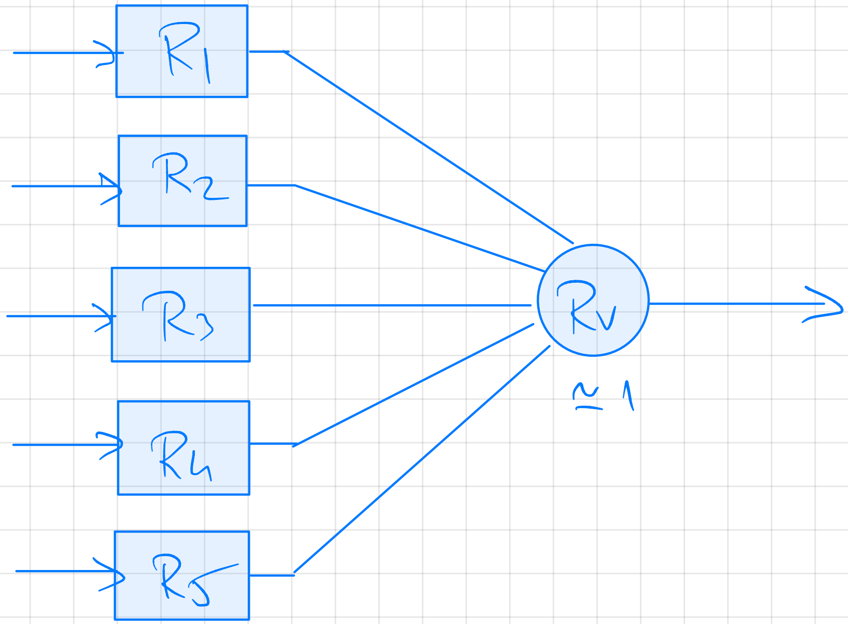
$t = \frac{\ln 2}{\lambda} = 0{,}69\ MTBF$

$MTBF_{2/3} = \int_0^\infty R_{2/3}(t)\,dt = \int_0^\infty \left(3e^{-2\lambda t} - 2e^{-3\lambda t}\right)dt = 3\int_0^\infty e^{-2\lambda t}\,dt - 2\int_0^\infty e^{-3\lambda t}\,dt =$

$= 3 \cdot \frac{1}{2\lambda} - 2 \cdot \frac{1}{3\lambda} = \frac{5}{6} \cdot \frac{1}{\lambda} = \frac{5}{6}\ MTBF < MTBF\ !$

$MTBF_{2/3} = \frac{5}{6}\ MTBF$

Structuri cu votare 3/5



$R_{3/5} = ?$

$R_1 = R_2 \ldots R_5 = R$

$$R_{3/5} = R^5 + C_5^4 (1-R) R^4 + C_5^3 (1-R)^2 R^3 =$$

$$= 6R^5 - 15R^4 + 10R^3$$

$R_{3/5} > R$?

$6R^5 - 15R^4 + 10R^3 > R$ pt. $R > 1/2$

$R = e^{-\lambda t} \Rightarrow R_{3/5} = 6e^{-5\lambda t} - 15e^{-4\lambda t} + 10e^{-3\lambda t}$

$$MTBF_{3/5} = \int_0^\infty R_{3/5}(t)\,dt = 6 \cdot \frac{1}{5\lambda} - 15 \cdot \frac{1}{4\lambda} + 10 \cdot \frac{1}{3\lambda} = \frac{47}{60} MTBF < MTBF_{2/3}$$

$MTBF_{3/5} < MTBF_{2/3} < MTBF$

Caz general: structură cu vot majoritar n din 2n-1

$$R_{n|2n-1} = \sum_{i=n}^{2n-1} C_{2n-1}^{i} (1-R)^{2n-1-i} R^{i}$$

$$R(t) = e^{-\lambda t}$$

$$R_{n|2n-1}(t) = \sum_{i=n}^{2n-1} C_{2n-1}^{i} \left(1 - e^{-\lambda t}\right)^{2n-1-i} e^{-\lambda i t}$$

$$MTBF_{n|2n-1} = \int_{0}^{\infty} R_{n|2n-1}(t)\,dt = \dots = \frac{1}{\lambda} \sum_{i=n}^{2n-1} \frac{1}{i}$$

Dacă $n \to \infty \Rightarrow \sum_{i=n}^{2n-1} \frac{1}{i} \to \ln(2) \Rightarrow MTBF_{n|2n-1} \to 0{,}69\, MTBF$

Scheme cu back-up (rezervare)

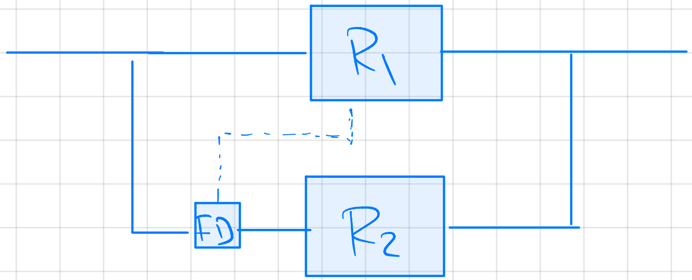
- Cold spare
- Warm spare

Un element de rezervă rece (cold spare)

R_1 - element de bază

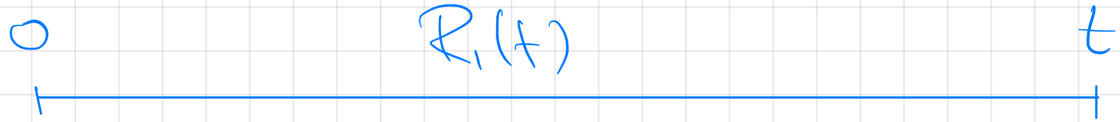
R_2 - rezervă

FD - Failure Detector



$R_{TOTAL} = ?$

Cazul 1:



Cazul 2:

Probabilitatea ca modulul de bază să se defecteze la τ este $f(\tau)$

$$f(\tau) = \frac{dF(\tau)}{d\tau} = -\frac{dR(\tau)}{d\tau}$$

Prob. ca 1 să se defecteze la τ și 2 să preia funcționarea este: $f_1(\tau)\cdot R_2(t-\tau)$

Dar τ poate să fie oricând în $[0,t]$ =>

$$R_{CAZ2}(t) = \int_0^t f_1(\tau) R_2(t-\tau)\,d\tau = \int_0^t -\frac{dR_1(\tau)}{d\tau} R_2(t-\tau)\,d\tau$$

$$R_{TOTAL} = R_{CAZ1} + R_{CAZ2} = R_1(t) + \int_0^t -\frac{dR_1(\tau)}{d\tau} R_2(t-\tau)\,d\tau$$

Dacă $R_1(t) = e^{-\lambda_1 t}$ și $R_2(t) = e^{-\lambda_2 t}$

$$-\frac{dR_1(t)}{dt} = \lambda_1 e^{-\lambda_1 t}$$

$$R_{TOTAL}(t) = e^{-\lambda_1 t} + \int_0^t \lambda_1 e^{-\lambda_1 \tau} e^{-\lambda_2(t-\tau)}\,d\tau = e^{-\lambda_1 t} + \lambda_1 e^{-\lambda_2 t}\int_0^t e^{-(\lambda_1-\lambda_2)\tau}\,d\tau$$

$$= e^{-\lambda_1 t} + \lambda_1 e^{-\lambda_2 t}\frac{1}{-(\lambda_1-\lambda_2)} e^{-(\lambda_1-\lambda_2)\tau}\Big|_0^t = e^{-\lambda_1 t} - \frac{\lambda_1}{\lambda_1-\lambda_2} e^{-\lambda_2 t}\left(e^{-(\lambda_1-\lambda_2)t} - 1\right) = \dots =$$

$$= \frac{\lambda_2}{\lambda_2 - \lambda_1} e^{-\lambda_1 t} - \frac{\lambda_1}{\lambda_2 - \lambda_1} e^{-\lambda_2 t}$$

$$R_{TOTAL}(t) = \frac{\lambda_2}{\lambda_2 - \lambda_1} e^{-\lambda_1 t} - \frac{\lambda_1}{\lambda_2 - \lambda_1} e^{-\lambda_2 t}$$

$$MTBF_{TOTAL} = \int_0^\infty R_{TOTAL}(t)\,dt = \frac{\lambda_2}{\lambda_2 - \lambda_1} \cdot \frac{1}{\lambda_1} - \frac{\lambda_1}{\lambda_2 - \lambda_1} \cdot \frac{1}{\lambda_2} = \frac{\lambda_2^2 - \lambda_1^2}{\lambda_1 \lambda_2 (\cancel{\lambda_2 - \lambda_1})} =$$

$$= \frac{\lambda_1 + \lambda_2}{\lambda_1 \lambda_2} = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} = MTBF_1 + MTBF_2$$

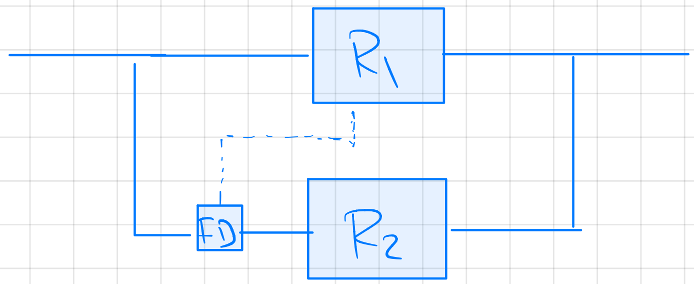
Dove $R_1(t) = R_2(t) = e^{-\lambda t}$

$$R_{TOTAL}(t) = e^{-\lambda t} + \int_0^t \lambda e^{-\lambda \tau} e^{-\lambda(t-\tau)} d\tau = e^{-\lambda t} + \lambda e^{-\lambda t} \int_0^t d\tau =$$

$$= e^{-\lambda t} + \lambda t \cdot e^{-\lambda t} = e^{-\lambda t}(1 + \lambda t)$$

$$MTBF_{TOTAL} = \int_0^\infty R_{TOTAL}(t)\,dt = \ldots = 2 \cdot \frac{1}{\lambda} = 2 MTBF$$

Un singur element de back-up cu warm sparing



Cazul 1: 0 —— $R_1(t)$ —— t

Cazul 2: 0 —— $R_1(t)$, $R_{2u}(t)$ —— τ —— $R_2(t-\tau)$ —— t

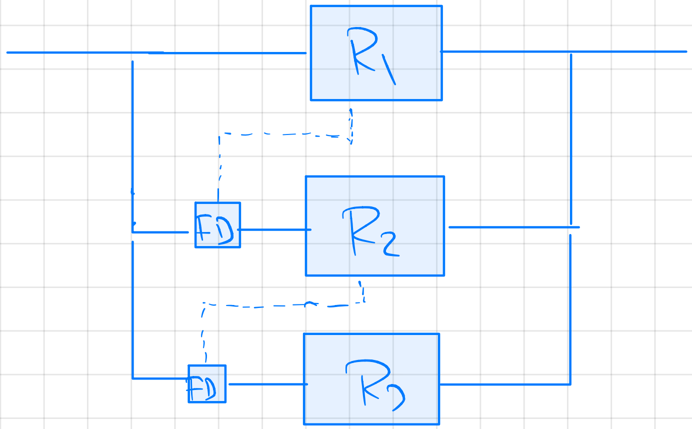
$$R_{TOTAL} = R_{CAZ1} + R_{CAZ2} = R_1(t) + \int_0^t -\frac{dR_1(\tau)}{d\tau} \cdot R_{2u}(\tau) \cdot R_2(t-\tau)\, d\tau$$

$R_1(t) = e^{-\lambda_1 t}$, $R_2(t) = e^{-\lambda_2 t}$ și $R_{2u}(t) = e^{-\lambda_{2u} t}$

$$R_{TOTAL}(t) = e^{-\lambda_1 t} + \frac{\lambda_1}{\lambda_1 + \lambda_{2u} - \lambda_2}\left(e^{-\lambda_2 t} - e^{-(\lambda_1+\lambda_{2u})t}\right)$$

$$MTBF_{TOTAL} = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} \cdot \frac{\lambda_1}{\lambda_1 + \lambda_{2u}} = MTBF_1 + MTBF_2 \cdot \frac{MTBF_{2u}}{MTBF_1 + MTBF_{2u}}$$

Două elemente de back-up, cold sparing



$$R_{12}(t) = R_1(t) + \int_0^t -\frac{dR_1(\tau)}{d\tau} R_2(t-\tau)\,d\tau$$

$$R_{123}(t) = R_{12}(t) + \int_0^t -\frac{dR_{12}(\tau)}{d\tau} R_3(t-\tau)\,d\tau$$

Dacă $R_1 = R_2 = R_3 = e^{-\lambda t}$

$$R_{123}(t) = e^{-\lambda t}\left(1 + \lambda t + \frac{\lambda^2 t^2}{2}\right)$$

$$MTBF_{123} = 3 \cdot \frac{1}{\lambda} = 3\,MTBF$$

Caz general: n-1 elemente de back-up

$$R_{usp} = e^{-\lambda t}\left(1 + \lambda t + \frac{\lambda^2 t^2}{2} + \frac{\lambda^3 t^3}{6} + \dots \frac{\lambda^n t^n}{n!}\right)$$

$$= e^{-\lambda t} \sum_{i=0}^{n} \frac{(\lambda t)^i}{i!}$$

$$MTBF = n \cdot \frac{1}{\lambda}$$

Caz special: dacă avem ∞ backup ?

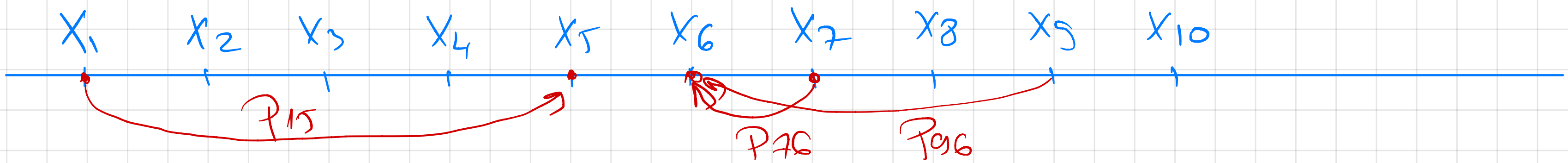
$$R_{\infty sp} = e^{-\lambda t} \sum_{i=0}^{\infty} \frac{(\lambda t)^i}{i!} = e^{-\lambda t} \cdot e^{\lambda t} = e^0 = 1$$

Modele Markov

- Starea sistemului: $x_1, x_2 \ldots x_n$
- timpul de observare: $t_1, t_2 \ldots t_n$

Lanturi Markov

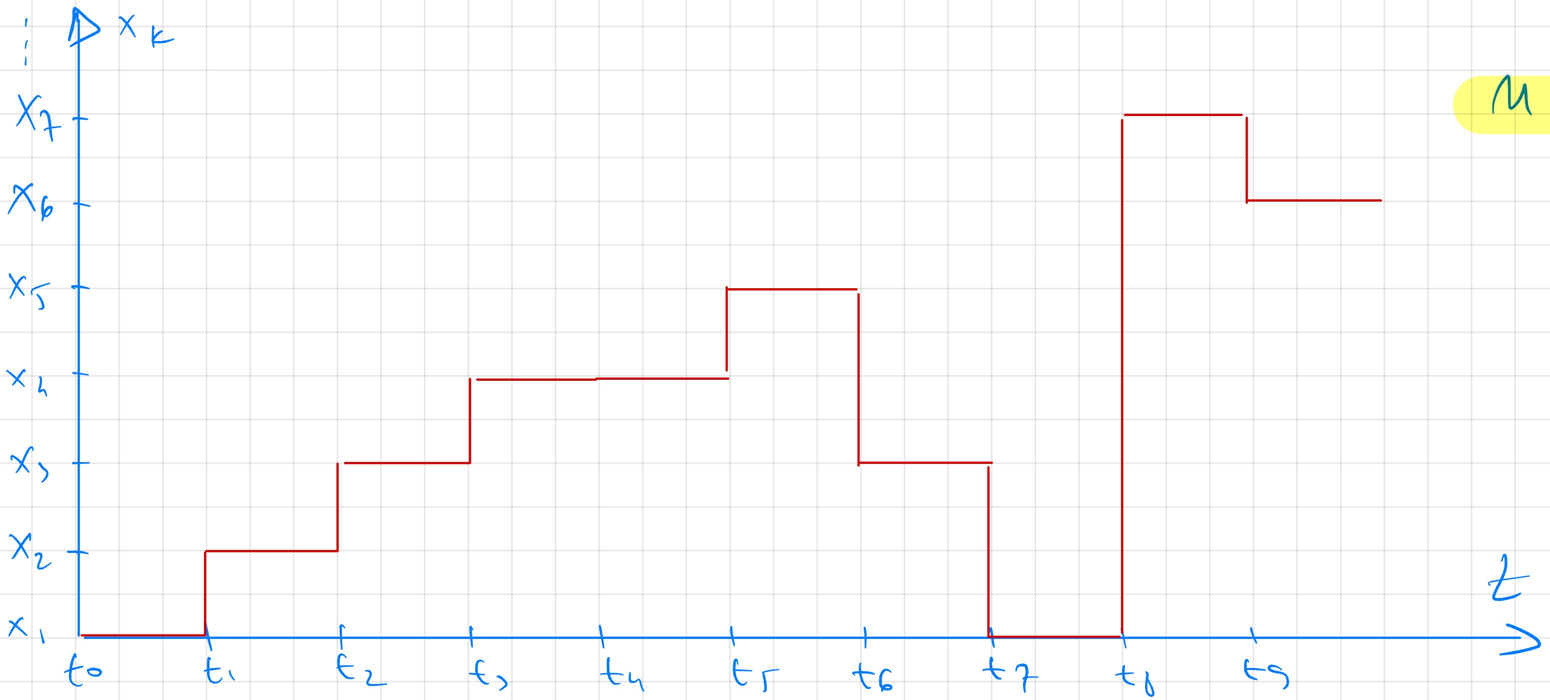
- starile prin care trece sistemul sunt discrete
- timpul de observatie este discretizat



$p_i(k) = P(S_k = x_i)$ - probabilitatea ca sistemul sa fie in starea x_i

$$\sum_{i=1}^{n} p_i(k) = 1$$

$p_{ij} = P(S_i \text{ la } t \mid S_j \text{ la } t+\delta t)$ -> probabilitatea de tranzitie din S_i in S_j



n stări → n × n prob. de tranziție

$$\begin{pmatrix} p_{11} & p_{12} & p_{13} & \dots & p_{1n} \\ p_{21} & p_{22} & p_{23} & \dots & p_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ p_{n1} & p_{n2} & p_{n3} & \dots & p_{nn} \end{pmatrix} = A$$

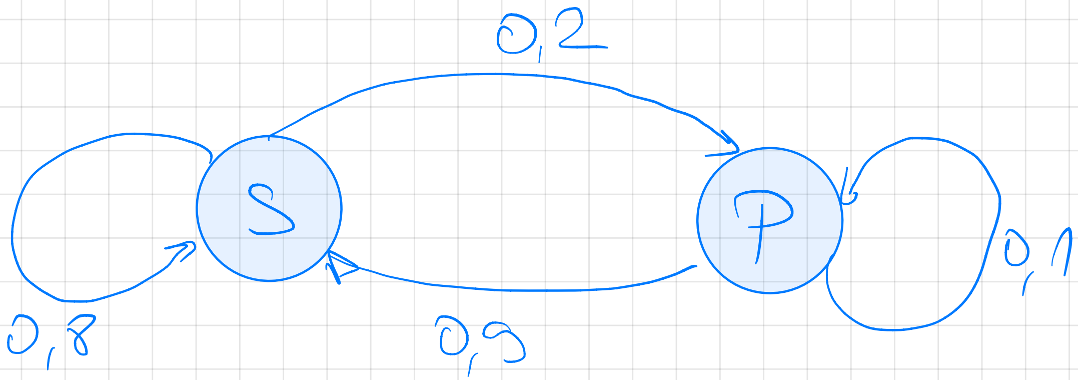
$$\left[P_1(t) \quad P_2(t) \quad \dots \quad P_n(t)\right] \cdot A = \left[P_1(t+\Delta t) \quad P_2(t+\Delta t) \quad \dots \quad P_n(t+\Delta t)\right]$$

Exemplu: starea vremii

Două stări posibile: soare (S), ploaie (P)

S
S
S
P
S
P
S
S
P
P
⋮

$$\begin{array}{c|cc} & S & P \\ \hline S & 0{,}8 & 0{,}2 \\ P & 0{,}9 & 0{,}1 \end{array} \qquad \begin{pmatrix} 0{,}8 & 0{,}2 \\ 0{,}9 & 0{,}1 \end{pmatrix} = A$$



$$\underset{\text{azi}}{\overset{\begin{array}{cc} S & P \end{array}}{\begin{pmatrix} 1 & 0 \end{pmatrix}}} \begin{pmatrix} 0{,}8 & 0{,}2 \\ 0{,}9 & 0{,}1 \end{pmatrix} = \underset{\text{mâine}}{\begin{pmatrix} 0{,}8 & 0{,}2 \end{pmatrix}} \cdot A = \underset{\text{poimâine}}{\begin{pmatrix} 0{,}82 & 0{,}18 \end{pmatrix}}$$

În general:

$$\boxed{[P(t)] \cdot A^n = [P(t + n\Delta t)]}$$

Cum este vremea după o perioadă lungă de timp? (steady-state)

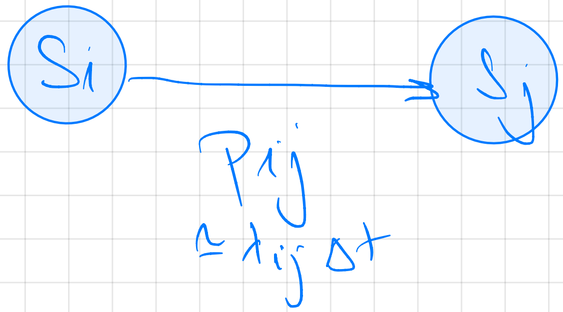
$$q \cdot A = q \Rightarrow q \cdot A = q \cdot I_2 \Rightarrow q(A - I_2) = (0 \quad 0)$$

$$(q_1 \quad q_2) \cdot \begin{pmatrix} -0,2 & 0,2 \\ 0,5 & -0,5 \end{pmatrix} = (0 \quad 0) \Rightarrow -0,2 \cdot q_1 + 0,5 \cdot q_2 = 0 \Rightarrow$$

$$\begin{cases} 2q_1 - 5q_2 = 0 \\ q_1 + q_2 = 1 \mid \cdot 5 \end{cases} \Rightarrow 11 q_1 = 5 \Rightarrow q_1 = \frac{5}{11}, \quad q_2 = \frac{2}{11}$$

$$\begin{pmatrix} \frac{9}{11} & \frac{2}{11} \end{pmatrix} = (0,81 \quad 0,18) \longrightarrow \text{este mai mult însorit}$$

Procese Markov



nu folosim p_{ij} ci λ_{ij}, unde

λ_{ij} este densitate de probabilitate de tranziție din S_i în S_j

$$\lambda_{ij} = \lim_{\Delta t \to 0} \frac{P\left(X(t)=S_i \mid X(t+\Delta t)=S_j\right)}{\Delta t}$$

Dacă $\Delta t \to 0 \Rightarrow p_{ij} \simeq \lambda_{ij} \cdot \Delta t$

$$[P(t)] = [P_1(t) \;\; P_2(t) \;\; \dots \;\; P_n(t)]$$

Se păstrează proprietatea:

$$[P(t+\Delta t)] = [P(t)] \cdot A$$

$$A = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{pmatrix} \approx \begin{pmatrix} \lambda_{11}\Delta t & \lambda_{12}\Delta t & \cdots & \lambda_{1n}\Delta t \\ \lambda_{21}\Delta t & \lambda_{22}\Delta t & \cdots & \lambda_{2n}\Delta t \\ \vdots & \vdots & & \vdots \\ \lambda_{n1}\Delta t & \lambda_{n2}\Delta t & \cdots & \lambda_{nn}\Delta t \end{pmatrix} =$$

$$= \begin{pmatrix} 1 - \sum_{i=2}^{n} \lambda_{1i}\Delta t & \lambda_{12}\Delta t & \cdots & \lambda_{1n}\Delta t \\ \lambda_{21}\Delta t & 1 - \sum_{\substack{i=1 \\ i\neq 2}}^{n} \lambda_{2i}\Delta t & \cdots & \lambda_{2n}\Delta t \\ \vdots & \vdots & & \vdots \\ \lambda_{n1}\Delta t & \lambda_{n2}\Delta t & \cdots & 1 - \sum_{i=1}^{n-1} \lambda_{ni}\Delta t \end{pmatrix}$$

$$[P(t+\Delta t)] = [P(t)] \cdot A$$

$$\begin{bmatrix} P_1(t) & P_2(t) & \dots & P_n(t) \end{bmatrix} \cdot \begin{pmatrix} 1-\sum_{i=2}^{n}\lambda_{1i}\Delta t & \lambda_{12}\Delta t & \dots & \lambda_{1n}\Delta t \\ \lambda_{21}\Delta t & 1-\sum_{\substack{i=1\\ i\neq 2}}^{n}\lambda_{2i}\Delta t & \dots & \lambda_{2n}\Delta t \\ \dots & \dots & \dots & \dots \\ \lambda_{n1}\Delta t & \lambda_{n2}\Delta t & \dots & 1-\sum_{i=1}^{n-1}\lambda_{ni}\Delta t \end{pmatrix}$$

$$= \begin{bmatrix} P_1(t+\Delta t) & P_2(t+\Delta t) & \dots & P_n(t+\Delta t) \end{bmatrix}$$

$$P_1(t+\Delta t) = P_1(t)\left(1-\sum_{i=2}^{n}\lambda_{1i}\Delta t\right) + P_2\lambda_{21}\Delta t + \dots P_n(t)\lambda_{n1}\Delta t \Leftrightarrow$$

$$P_1(t+\Delta t) - P_1(t) = -P_1(t)\sum_{i=2}^{n}\lambda_{1i}\Delta t + P_2(t)\lambda_{21}\Delta t + \dots P_n(t)\lambda_{n1}\Delta t \quad / : \Delta t \Rightarrow$$

$$\frac{P_1(t+\Delta t)-P_1(t)}{\Delta t} = \sum_{i=2}^{n} P_i(t)\lambda_{i1} - P_1(t)\sum_{i=2}^{n}\lambda_{1i} \Rightarrow$$

$$\lim_{\Delta t \to 0} \frac{P_1(t+\Delta t) - P_1(t)}{\Delta t} = \sum_{i=2}^{n} P_i(t)\lambda_{i1} - P_1(t)\sum_{i=2}^{n} \lambda_{1i}$$

$$\frac{dP_1(t)}{dt} = \sum_{i=2}^{n} P_i(t)\lambda_{i1} - P_1(t)\sum_{i=2}^{n} \lambda_{1i}$$

În general:

$$\boxed{\frac{dP_j(t)}{dt} = \sum_{\substack{i=1 \\ i\neq j}}^{n} P_i(t)\lambda_{ij} - P_j(t)\sum_{\substack{i=1 \\ i\neq j}}^{n} \lambda_{ji}}$$

Sistemul de ecuații Chapman-Kolmogorov

Forma matriceală a sistemului C-K:

$$[P(t)] \cdot A^{*} = [P'(t)]$$

, unde $A^* = \begin{pmatrix} -\sum_{i=2}^{n} \lambda_{1i} & \lambda_{12} & \dots & \lambda_{1n} \\ \lambda_{21} & -\sum_{\substack{i=1 \\ i\neq 2}}^{n} \lambda_{2i} & \dots & \lambda_{2n} \\ \dots & \dots & \dots & \dots \\ \lambda_{n1} & \lambda_{n2} & \dots & -\sum_{i=1}^{n-1} \lambda_{ni} \end{pmatrix}$

Soluția generală pentru sistemul de ecuații C-K :

Rescriem ecuația $[P_1(t)\ P_2(t)\ \dots\ P_n(t)] \cdot A^* = \left[\frac{dP_1(t)}{dt}\ \frac{dP_2(t)}{dt}\ \dots\ \frac{dP_n(t)}{dt}\right]$

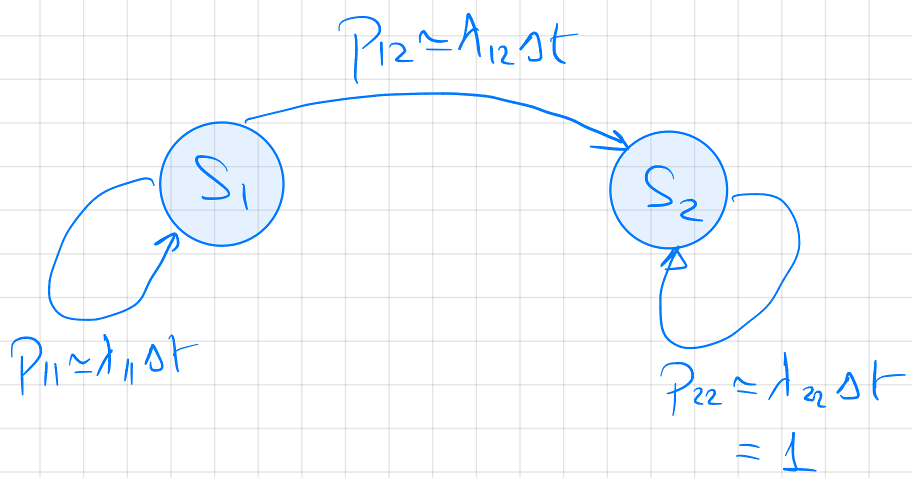
ca : $\begin{pmatrix} -\sum_{i=2}^{n} \lambda_{1i} & \lambda_{21} & \dots & \lambda_{n1} \\ \lambda_{12} & -\sum_{\substack{i=1 \\ i\neq 2}}^{n} \lambda_{2i} & \dots & \lambda_{n2} \\ \vdots & \vdots & & \vdots \\ \lambda_{1n} & \lambda_{2n} & & -\sum_{i=1}^{n-1} \lambda_{ni} \end{pmatrix} \cdot \begin{pmatrix} P_1(t) \\ P_2(t) \\ \vdots \\ P_n(t) \end{pmatrix} = \begin{pmatrix} P_1'(t) \\ P_2'(t) \\ \vdots \\ P_n'(t) \end{pmatrix} \Leftrightarrow M \cdot [P(t)] = [P'(t)]$

Soluția ecuației $M \cdot [P(t)] = [P'(t)]$ este: $[P(t)] = e^{Mt} \cdot [P(0)]$

Exemplu: sistem cu două stări

S_1: funcționare

S_2: defect



Fără reparare! (S_2 e stare de absorbție)

$$A = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} = \begin{pmatrix} \lambda_{11}\Delta t & \lambda_{12}\Delta t \\ 0 & 1 \end{pmatrix}$$

$$A^* = \begin{pmatrix} -\lambda_{12} & \lambda_{12} \\ 0 & 0 \end{pmatrix} \xRightarrow{CK} \begin{pmatrix} P_1(t) & P_2(t) \end{pmatrix} \cdot \begin{pmatrix} -\lambda_{12} & \lambda_{12} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} P_1'(t) & P_2'(t) \end{pmatrix} \Rightarrow$$

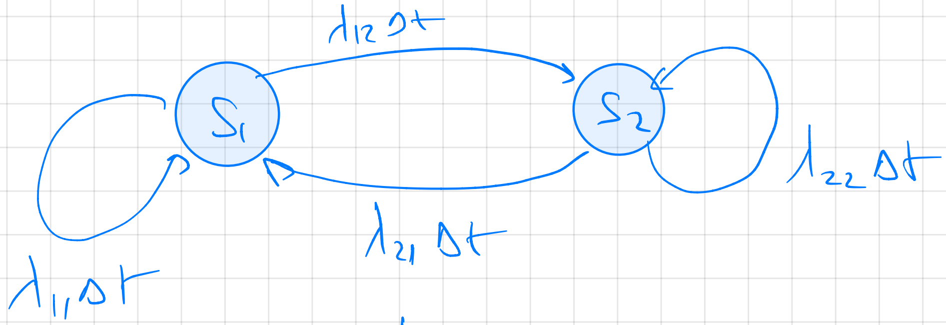
$$\Rightarrow \begin{cases} P_1'(t) = -\lambda_{12} P_1(t) \\ P_2'(t) = \lambda_{12} P_1(t) \end{cases} \Rightarrow \frac{dP_1(t)}{dt} = -\lambda_{12} P_1(t) \Rightarrow dP_1(t) \cdot \frac{1}{P_1(t)} = -\lambda_{12}\, dt \;\Big|\cdot\int \Rightarrow$$

$$\Rightarrow \ln(P_1(t)) = -\lambda_{12} t + c \Rightarrow P_1(t) = e^{-\lambda_{12} t + c}$$

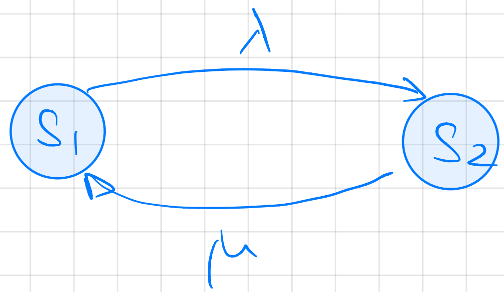
Dacă $P_1(0) = 1 \Rightarrow e^c = 1 \Rightarrow c = 0 \Rightarrow$ $P_1(t) = e^{-\lambda_{12} t} = R(t)$

$$P_2(t) = 1 - P_1(t) = 1 - e^{-\lambda_{12} t} = F(t)$$

Dacă adăugăm posibilitatea de reparare:



simplificare



$$\lambda_{12} = \lambda$$

$$\lambda_{21} = \mu$$

$$A^* = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}$$

$$\begin{pmatrix} P_1(t) & P_2(t) \end{pmatrix} \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix} = \begin{pmatrix} -\lambda P_1(t) + \mu P_2(t) & \lambda P_1(t) - \mu P_2(t) \end{pmatrix}$$

deci:

CK

$$\begin{cases} \dfrac{dP_1(t)}{dt} = -\lambda P_1(t) + \mu P_2(t) \\ \dfrac{dP_2(t)}{dt} = \lambda P_1(t) - \mu P_2(t) \end{cases}$$

$$\Rightarrow P_1'(t) = -\lambda P_1(t) + \mu(1 - P_1(t)) \Rightarrow$$

$$P_1'(t) + (\lambda + \mu) P_1(t) = \mu$$

dar $P_1(t) + P_2(t) = 1$

Flashback:

$$y'(t) + p(t)\, y(t) = g(t)$$ → ecuație diferențială liniară de gradul 1

Fie $\mu(t)$, cu proprietatea că $\boxed{\mu'(t) = \mu(t) \cdot p(t)}$

$$\mu'(t) = \mu(t) \cdot p(t) \Rightarrow \frac{\mu'(t)}{\mu(t)} = p(t) \Big| \cdot \int \Rightarrow \int \frac{\mu'(t)}{\mu(t)} = \int p(t)\,dt \Rightarrow \ln(\mu(t)) =$$

$$= \int p(t)\,dt + k \Rightarrow \boxed{\mu(t) = e^{\int p(t)\,dt + k}}$$

$$y'(t) + p(t)\, y(t) = g(t) \Big| \cdot \mu(t) \Rightarrow y'(t)\mu(t) + \mu(t) p(t)\, y(t) = g(t)\mu(t)$$

$$y'(t)\mu(t) + \mu'(t)\, y(t) = g(t)\mu(t) \Rightarrow \left(y(t)\mu(t)\right)' = g(t)\mu(t) \Rightarrow$$

$$y(t)\mu(t) = \int g(t)\mu(t)\,dt \Rightarrow y(t) = \frac{\int g(t)\mu(t)\,dt}{\mu(t)} \Rightarrow$$

$$\boxed{y(t) = \frac{\int e^{\int p(t)\,dt + k} g(t)\,dt}{e^{\int p(t)\,dt + k}}}$$

$$P'(t) + \underbrace{(\lambda+\mu)}_{p(t)} P(t) = \underbrace{\mu}_{g(t)}$$

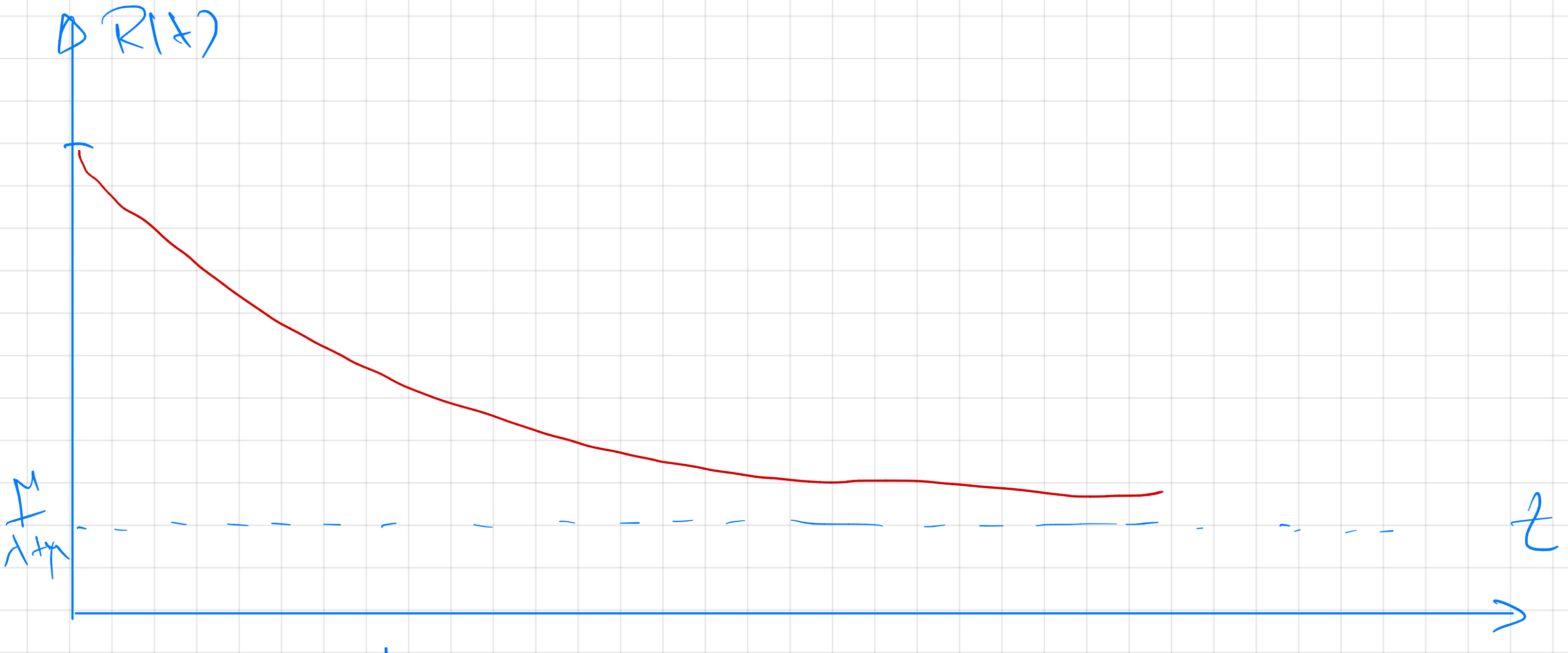
$$y(t) \qquad p(t) \qquad g(t)$$

$$P_1(t) = \frac{\int e^{\int(\lambda+\mu)dt} \mu \, dt}{e^{\int(\lambda+\mu)dt + k}} = \frac{\mu \int e^{(\lambda+\mu)t} + k}{e^{(\lambda+\mu)t}} = \frac{\mu \frac{1}{\lambda+\mu} e^{(\lambda+\mu)t} + k}{e^{(\lambda+\mu)t}} =$$

$$= \frac{\mu}{\lambda+\mu} + k \cdot e^{-(\lambda+\mu)t}$$

$$P_1(0) = 1 \Rightarrow \frac{\mu}{\lambda+\mu} + k = 1 \Rightarrow k = \frac{\lambda}{\lambda+\mu}$$

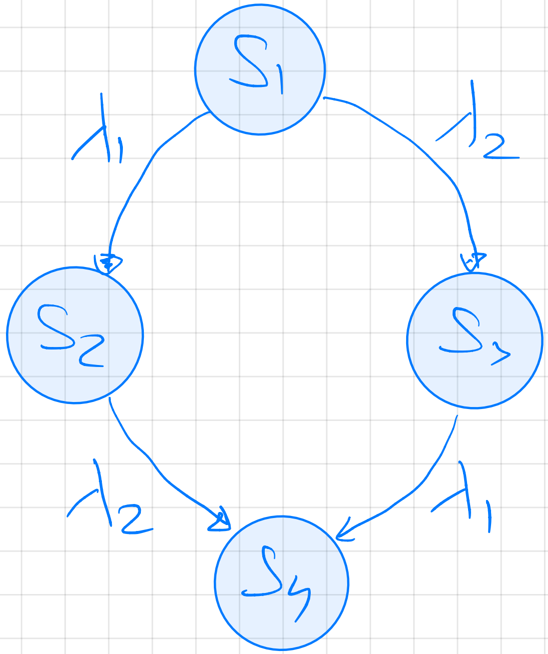
$$\boxed{P_1(t) = R(t) = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t}}$$



$$\frac{\mu}{\lambda+\mu} = \frac{\frac{1}{\lambda}}{\frac{1}{\mu}+\frac{1}{\lambda}} = \frac{MTBF}{MTR + MTBF} = A \text{ (disponibilità)}$$

Două componente - fără reparare

State	Comp. 1	Comp. 2
S_1	Operational	Op.
S_2	Defect	Op.
S_3	Op.	Defect
S_4	Defect	Defect



$$A^* = \begin{pmatrix} -\lambda_1-\lambda_2 & \lambda_1 & \lambda_2 & 0 \\ 0 & -\lambda_2 & 0 & \lambda_2 \\ 0 & 0 & -\lambda_1 & \lambda_1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$[P(t)]\, A^* = [P'(t)]$$

$$P_1'(t) = -(\lambda_1+\lambda_2)P_1(t) \Rightarrow \frac{dP_1(t)}{dt} = -(\lambda_1+\lambda_2)P_1(t) \Rightarrow \frac{1}{P_1(t)}\, dP_1(t) = -(\lambda_1+\lambda_2)dt$$

$$\int \frac{1}{P_1(t)}\, dP_1(t) = \int -(\lambda_1+\lambda_2)dt \Rightarrow \ln(P_1(t)) = -(\lambda_1+\lambda_2)t + C \Rightarrow$$

$$P_1(t) = e^{-(\lambda_1+\lambda_2)t + c}$$

$$P_1(0) = 1 \Rightarrow e^{-(\lambda_1+\lambda_2)0 + c} = 1 \Rightarrow e^{c} = 1 \Rightarrow c = 0$$

$$P_1(t) = e^{-(\lambda_1+\lambda_2)t}$$

$$P_2'(t) = \lambda_1 P_1(t) - \lambda_2 P_2(t) = \lambda_1 e^{-(\lambda_1+\lambda_2)t} - \lambda_2 P_2(t) \Rightarrow$$

$$\underbrace{P_2'(t)} + \underbrace{\lambda_2} P_2(t) = \underbrace{\lambda_1 e^{-(\lambda_1+\lambda_2)t}}$$

$$y'(t) + p(t)\, y(t) = g(t) \qquad y(t) = \frac{\int e^{\int p(t)dt} g(t)dt + k}{e^{\int p(t)dt}}$$

$$P_2(t) = \frac{\int e^{\int \lambda_2 dt} \cdot \lambda_1 e^{-(\lambda_1+\lambda_2)t} dt + k}{e^{\int \lambda_2 dt}} = \frac{\lambda_1 \int \cancel{e^{\lambda_2 t}} e^{-\lambda_1 t} \cdot \cancel{e^{-\lambda_2 t}} dt + k}{e^{\lambda_2 t}} =$$

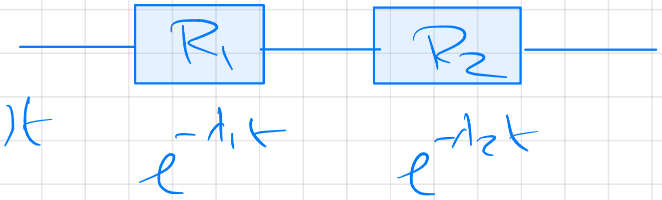
$$= \lambda_1 e^{-\lambda_2 t}\left(\int e^{-\lambda_1 t} dt + k\right) = \lambda_1 e^{-\lambda_2 t}\left(\frac{e^{-\lambda_1 t}}{-\lambda_1} + k\right) = k\lambda_1 e^{-\lambda_2 t} - e^{-(\lambda_1+\lambda_2)t}$$

$$P_2(0) = 0 \Rightarrow k\lambda_1 e^{0} - e^{0} = 0 \Rightarrow k\lambda_1 - 1 = 0 \Rightarrow k = \frac{1}{\lambda_1} \Rightarrow P_2(t) = e^{-\lambda_2 t} - e^{-(\lambda_1+\lambda_2)t}$$

$$P_3'(t) = \lambda_2 P_1(t) - \lambda_1 P_3(t) \Rightarrow \dots \Rightarrow P_3(t) = e^{-\lambda_1 t} - e^{-(\lambda_1+\lambda_2)t}$$

$$P_4(t) = 1 - P_1(t) - P_2(t) - P_3(t)$$

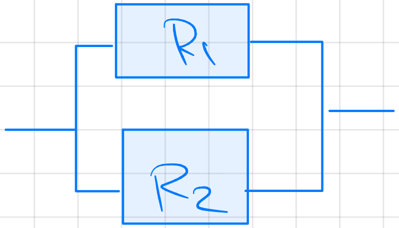
Sistem cu două module în serie:



$$R_{serie} = R_1 \cdot R_2 = e^{-\lambda_1 t} \cdot e^{-\lambda_2 t} = e^{-(\lambda_1+\lambda_2)t}$$

S_1 = modulele în serie, $R_{serie} = P_1(t)$

Sistem cu două module în paralel:

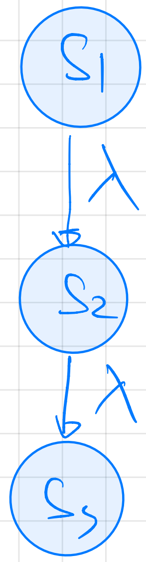


$$R_{paralel} = 1 - (1-R_1)(1-R_2) = R_1 + R_2 - R_1 R_2 =$$
$$= e^{-\lambda_1 t} + e^{-\lambda_2 t} - e^{-(\lambda_1+\lambda_2)t}$$

$$P_1(t) + P_2(t) + P_3(t) = \cancel{e^{-(\lambda_1+\lambda_2)t}} + e^{-\lambda_2 t} - \cancel{e^{-(\lambda_1+\lambda_2)t}} + e^{-\lambda_1 t} - e^{-(\lambda_1+\lambda_2)t} =$$
$$= e^{-\lambda_1 t} + e^{-\lambda_2 t} - e^{-(\lambda_1+\lambda_2)t}$$

Sistem cu un modul de back-up

Stări	Baza	Back-up
S_1	Op.	Stdby.
S_2	Fail	Op.
S_3	Fail	Fail



P.p. că baza și back-up sunt identice
$\lambda_1 = \lambda_2 = \lambda$

$$A^* = \begin{pmatrix} -\lambda & \lambda & 0 \\ 0 & -\lambda & \lambda \\ 0 & 0 & 0 \end{pmatrix}$$

$P_1'(t) = -\lambda P_1(t) \Rightarrow \dots \Rightarrow \boxed{P_1(t) = e^{-\lambda t}}$

$P_2'(t) = \lambda P_1(t) - \lambda P_2(t) = \lambda e^{-\lambda t} - \lambda P_2(t)$

$P_2'(t) + \underbrace{\lambda}_{p(t)} P_2(t) = \underbrace{\lambda e^{-\lambda t}}_{g(t)}$

$$P_2(t) = \frac{\int e^{\int \lambda dt} \lambda e^{-\lambda t} dt + k}{e^{\int \lambda dt}} =$$

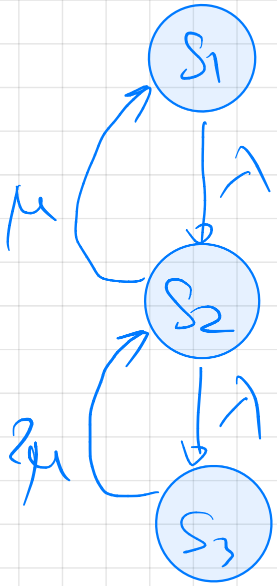
$$= e^{-\lambda t}\left(\lambda \int \cancel{e^{\lambda t}} \cancel{e^{-\lambda t}} dt + k\right) = e^{-\lambda t}(\lambda t + k)$$

$P_2(0) = 0 \Rightarrow e^{-\lambda \cdot 0}(\lambda \cdot 0 + k) = 0 \Rightarrow k = 0 \Rightarrow \boxed{P_2(t) = \lambda t e^{-\lambda t}}$

$$R(t) = P_1(t) + P_2(t) = e^{-\lambda t} + \lambda t\, e^{-\lambda t} = e^{-\lambda t}(1+\lambda t)$$

Sistem cu un modul de bock-up și reparare

$$A^* = \begin{pmatrix} -\lambda & \lambda & 0 \\ \mu & -\lambda-\mu & \lambda \\ 0 & 2\mu & -2\mu \end{pmatrix}$$



$\lambda = 10/an$

$\mu = 5/an$

$$\begin{cases} P_1'(t) = -\lambda P_1(t) + \mu P_2(t) \\ P_2'(t) = \lambda P_1(t) - (\lambda+\mu) P_2(t) + 2\mu P_3(t) \\ P_3'(t) = \lambda P_2(t) - 2\mu P_3(t) \end{cases}$$

$$\underbrace{\begin{pmatrix} -\lambda & \mu & 0 \\ \lambda & -(\lambda+\mu) & 2\mu \\ 0 & \lambda & -2\mu \end{pmatrix}}_{M} \begin{pmatrix} P_1(t) \\ P_2(t) \\ P_3(t) \end{pmatrix} = \begin{pmatrix} P_1'(t) \\ P_2'(t) \\ P_3'(t) \end{pmatrix} \Leftrightarrow M[P(t)] = [P'(t)]$$ cu soluția:

$$e^{Mt}[P(0)] = [P(t)]$$

La steady-state $(t \to \infty) \Rightarrow \frac{dP(t)}{dt} \simeq 0$ și $P(t) = P = ct$

$$\begin{cases} 0 = -\lambda \Pi_1 + \mu \Pi_2 \\ 0 = \lambda \Pi_1 - (\lambda + \mu) \Pi_2 + 2\mu \Pi_3 \\ 0 = \lambda \Pi_2 - 2\mu \Pi_3 \end{cases}$$

$$\Pi_1 + \Pi_2 + \Pi_3 = 1 \Rightarrow \Pi_3 = 1 - \Pi_1 - \Pi_2$$

Disponibilitatea sistemului este $A = \Pi_1 + \Pi_2$

$$\lambda \Pi_2 - 2\mu(1 - \Pi_1 - \Pi_2) = 0 \Rightarrow \lambda \Pi_2 - 2\mu + 2\mu \Pi_1 + 2\mu \Pi_2 = 0 \Rightarrow$$

$$\Rightarrow (\lambda + 2\mu) \Pi_2 + 2\mu \Pi_1 = 2\mu$$

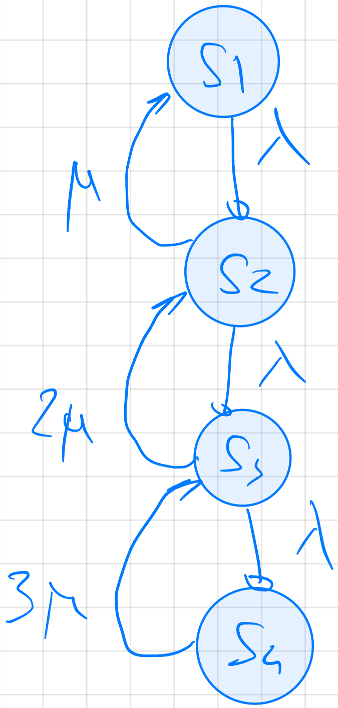
$$\lambda \Pi_1 = \mu \Pi_2 \Rightarrow \Pi_2 = \frac{\lambda}{\mu} \Pi_1 \Rightarrow \frac{\lambda(\lambda + 2\mu)}{\mu} \Pi_1 + 2\mu \Pi_1 = 2\mu \Rightarrow$$

$$\Rightarrow \Pi_1 \frac{\lambda^2 + 2\mu\lambda + 2\mu^2}{\mu} = 2\mu \Rightarrow \Pi_1 = \frac{2\mu^2}{\lambda^2 + 2\mu\lambda + 2\mu^2}, \quad \Pi_2 = \frac{2\mu\lambda}{\lambda^2 + 2\mu\lambda + 2\mu^2}$$

$$A = \Pi_1 + \Pi_2 = \frac{2\mu^2 + 2\mu\lambda}{\lambda^2 + 2\mu\lambda + 2\mu^2} = \frac{2 \cdot 25 + 2 \cdot 5 \cdot 10}{100 + 2 \cdot 5 \cdot 10 + 2 \cdot 25} = \frac{150}{250} = 60\%$$

Sistem cu două module de back-up și reparare

Stare	Baza	Back-up 1	Back-up 2
S_1	OK	Stdby.	Stdby.
S_2	Fail	OK	Stdby.
S_3	Fail	Fail	OK
S_4	Fail	Fail	Fail



$$A^* = \begin{pmatrix} -\lambda & \lambda & 0 & 0 \\ \mu & -(\lambda+\mu) & \lambda & 0 \\ 0 & 2\mu & -(\lambda+2\mu) & \lambda \\ 0 & 0 & 3\mu & -3\mu \end{pmatrix}$$

$$\begin{cases} P_1'(t) = -\lambda P_1(t) + \mu P_2(t) \\ P_2'(t) = \lambda P_1(t) - (\lambda+\mu) P_2(t) + 2\mu P_3(t) \\ P_3'(t) = \lambda P_2(t) - (\lambda+2\mu) P_3(t) + 3\mu P_4(t) \\ P_4'(t) = \lambda P_3(t) - 3\mu P_4(t) \end{cases} \xrightarrow{\text{la limită}} \begin{cases} -\lambda \Pi_1 + \mu \Pi_2 = 0 \\ \lambda \Pi_1 - (\lambda+\mu)\Pi_2 + 2\mu \Pi_3 = 0 \\ \lambda \Pi_2 - (\lambda+2\mu)\Pi_3 + 3\mu \Pi_4 = 0 \\ \lambda \Pi_3 - 3\mu \Pi_4 = 0 \end{cases}$$

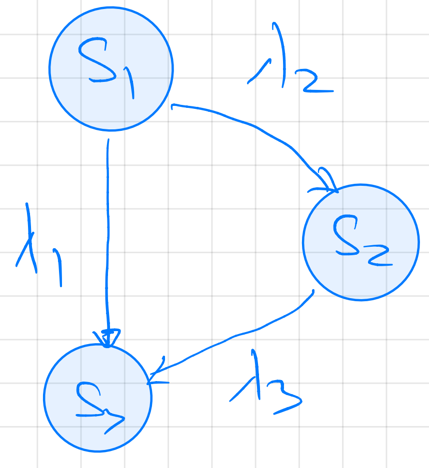
$$A = \Pi_1 + \Pi_2 + \Pi_3 = \ldots$$

Sistem cu funcționare degradată

S_1 : stare de funcționare completă

S_2 : stare de funcționare degradată

S_3 : defect



$$A^* = \begin{pmatrix} -\lambda_1-\lambda_2 & \lambda_2 & \lambda_1 \\ 0 & -\lambda_3 & \lambda_3 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{cases} P_1'(t) = -(\lambda_1+\lambda_2)P_1(t) \\ P_2'(t) = \lambda_2 P_1(t) - \lambda_3 P_2(t) \\ P_3'(t) = \lambda_1 P_1(t) + \lambda_3 P_2(t) \end{cases}$$

$$P_1(t) = e^{-(\lambda_1+\lambda_2)t}$$

$$P_2'(t) + \underbrace{\lambda_3}_{p(t)} P_2(t) = \underbrace{\lambda_2 e^{-(\lambda_1+\lambda_2)t}}_{g(t)} \Rightarrow P_2(t) = \frac{\lambda_2}{\lambda_1+\lambda_2-\lambda_3}\left(e^{-\lambda_3 t} - e^{-(\lambda_1+\lambda_2)t}\right)$$

$$R(t) = P_1(t) + P_2(t) = e^{-(\lambda_1+\lambda_2)t} + \frac{\lambda_2}{\lambda_1+\lambda_2-\lambda_3}\left(e^{-\lambda_3 t} - e^{-(\lambda_1+\lambda_2)t}\right)$$

$$MTBF = \int_0^\infty R(t)\,dt = \int_0^\infty \left(P_1(t) + P_2(t) \right) dt$$

Dați: $\lambda_1 = 5/an$, $\lambda_2 = 10/an$, $\lambda_3 = 2/an$

$$P_1(t) = e^{-(\lambda_1+\lambda_2)t} = e^{-15t}$$

$$P_2(t) = \frac{\lambda_2}{\lambda_1+\lambda_2-\lambda_3}\left(e^{-\lambda_3 t} - e^{-(\lambda_1+\lambda_2)t}\right) = \frac{10}{13}\left(e^{-2t} - e^{-15t}\right) = \frac{10}{13}\left(e^{-2t} - e^{-15t}\right)$$

$$MTBF = \int_0^\infty \left(e^{-15t} + \frac{10}{13}e^{-2t} - \frac{10}{13}e^{-15t} \right) dt = \frac{1}{15} + \frac{10}{13}\cdot\frac{1}{2} - \frac{10}{13}\cdot\frac{1}{15} = \frac{1}{15} + \frac{10}{26} - \frac{10}{195}$$

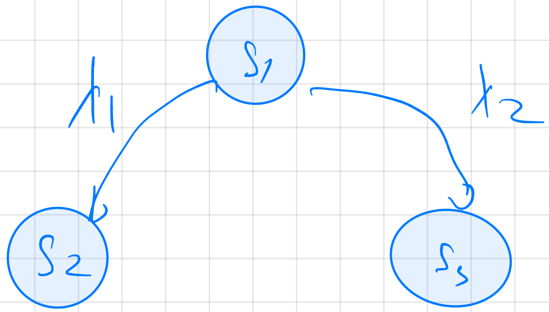
$$= 0{,}067 + 0{,}385 - 0{,}05 = 0{,}4 \text{ ani}$$

Sisteme three-state

S_1 : funcționare

S_2 : fail - open

S_3 : fail - short



$$A^* = \begin{pmatrix} -\lambda_1-\lambda_2 & \lambda_1 & \lambda_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{array}{l} P_1'(t) = -(\lambda_1+\lambda_2)P_1(t) \Rightarrow P_1(t) = e^{-(\lambda_1+\lambda_2)t} \\ P_2'(t) = \lambda_1 P_1(t) \\ P_3'(t) = \lambda_2 P_1(t) \end{array}$$

$$R(t) = P_1(t) = e^{-(\lambda_1+\lambda_2)t}$$

$\lambda_1 = 2/\text{oră}$

$\lambda_2 = 0{,}01/\text{oră}$

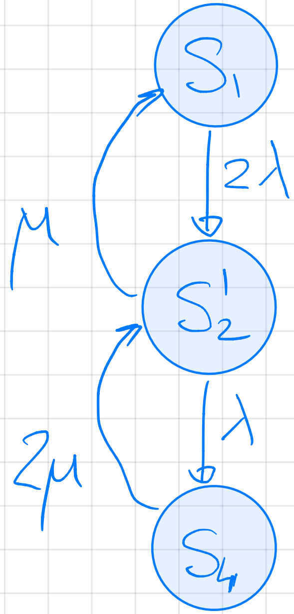
$$\Rightarrow R(t) = e^{-2{,}01t}$$

$$MTBF = \int_0^{\infty} R(t)\,dt = \int_0^{\infty} e^{-2{,}01t}\,dt = \frac{1}{2{,}01}\ \text{ore}$$

Două module identice - doi depanatori

Stare	Modul 1	Modul 2
S_1	OK	OK
S_2	Reparare	OK
S_3	OK	Reparare
S_4	Reparare	Reparare

S_2'



$$A^* = \begin{pmatrix} -2\lambda & 2\lambda & 0 \\ \mu & -\lambda-\mu & \lambda \\ 0 & 2\mu & -2\mu \end{pmatrix} \Rightarrow \begin{cases} P_1'(t) = -2\lambda P_1(t) + \mu P_2(t) \\ P_2'(t) = 2\lambda P_1(t) - (\lambda+\mu) P_2(t) + 2\mu P_4(t) \\ P_4'(t) = \lambda P_2(t) - 2\mu P_4(t) \end{cases}$$

la limită: $\begin{cases} -2\lambda \Pi_1 + \mu \Pi_2 = 0 \\ 2\lambda \Pi_1 - (\lambda+\mu)\Pi_2 + 2\mu \Pi_4 = 0 \\ \lambda \Pi_2 - 2\mu \Pi_4 = 0 \end{cases}$

Și $\Pi_1 + \Pi_2 + \Pi_4 = 1 \Rightarrow \Pi_4 = 1 - \Pi_1 - \Pi_2 \Rightarrow$

$$\begin{cases} -2\lambda\Pi_1 + \mu\Pi_2 = 0 \\ \lambda\Pi_2 - 2\mu(1 - \Pi_1 - \Pi_2) = 0 \end{cases} \Rightarrow \begin{cases} \Pi_2 = \frac{2\lambda}{\mu}\Pi_1 \\ (\lambda+2\mu)\Pi_2 + 2\mu\Pi_1 = 2\mu \end{cases} \Rightarrow (\lambda+2\mu)\frac{2\lambda}{\mu}\Pi_1 + 2\mu\Pi_1 = 2\mu \Rightarrow$$

$$\Pi_1\left(2\mu + \frac{2\lambda}{\mu}(\lambda+2\mu)\right) = 2\mu \Rightarrow \Pi_1 \frac{2\mu^2 + 2\lambda^2 + 4\lambda\mu}{\mu} = 2\mu \Rightarrow$$

$$\Rightarrow \Pi_1 = \frac{\mu^2}{\mu^2 + 2\lambda\mu + \lambda^2} = \frac{\mu^2}{(\lambda+\mu)^2} = \left(\frac{\mu}{\lambda+\mu}\right)^2$$

$$\Pi_2 = \frac{2\lambda}{\mu}\Pi_1 = \frac{2\lambda}{\cancel{\mu}} \cdot \frac{\mu^{\cancel{2}}}{(\lambda+\mu)^2} = \frac{2\lambda\mu}{(\lambda+\mu)^2}$$

Disponibilitatea sistemului: $A = \Pi_1 + \Pi_2 = \dfrac{\mu^2 + 2\lambda\mu}{(\lambda+\mu)^2} =$

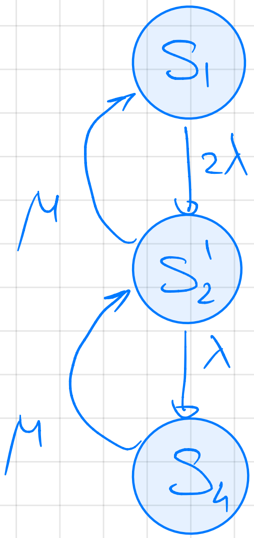
$$= \frac{\mu^2 + 2\lambda\mu + \lambda^2 - \lambda^2}{(\lambda+\mu)^2} = \frac{(\lambda+\mu)^2 - \lambda^2}{(\lambda+\mu)^2} = 1 - \left(\frac{\lambda}{\lambda+\mu}\right)^2 = 1 - \left(\frac{MTR}{MTR+MTBF}\right)^2$$

Dacă $\lambda = 1/an$ și $\mu = 10/an \Rightarrow A = 1 - \dfrac{1}{11^2} = 1 - \dfrac{1}{121} = 0{,}99 = 99\%$

Două module identice - un singur depanator

Stare	Modul 1	Modul 2
S_1	OK	OK
S_2	Reparare	OK
S_3	OK	Reparare
S_4	Fail / Rep.	Fail / Rep.

S_2'



$$A^* = \begin{pmatrix} -2\lambda & 2\lambda & 0 \\ \mu & -\lambda-\mu & \lambda \\ 0 & \mu & -\mu \end{pmatrix} \Rightarrow \begin{cases} P_1'(t) = -2\lambda P_1(t) + \mu P_2(t) \\ P_2'(t) = 2\lambda P_1(t) - (\lambda+\mu) P_2(t) + \mu P_4(t) \\ P_4'(t) = \lambda P_2(t) - \mu P_4(t) \end{cases}$$

La limită:
$$\begin{cases} -2\lambda \Pi_1 + \mu \Pi_2 = 0 & \Rightarrow \Pi_2 = \frac{2\lambda}{\mu}\Pi_1 \\ 2\lambda \Pi_1 - (\lambda+\mu)\Pi_2 + \mu \Pi_4 = 0 \\ \lambda \Pi_2 - \mu \Pi_4 = 0 \end{cases}$$

Si $\Pi_1 + \Pi_2 + \Pi_4 = 1 \Rightarrow \Pi_4 = 1 - \Pi_1 - \Pi_2$

$$\lambda \Pi_2 - \mu \Pi_4 = 0 \Rightarrow \lambda \Pi_2 - \mu(1 - \Pi_1 - \Pi_2) = 0 \Rightarrow (\lambda+\mu)\Pi_2 + \mu \Pi_1 = \mu \Rightarrow \frac{2\lambda}{\mu}(\lambda+\mu)\Pi_1 +$$

$$+\mu\Pi_1 = \mu \Rightarrow \Pi_1 \cdot \left(\frac{2\lambda}{\mu}(\lambda+\mu)+\mu\right) = \mu \Rightarrow \Pi_1 \frac{2\lambda^2+2\lambda\mu+\mu^2}{\mu} = \mu \Rightarrow$$

$$\Rightarrow \Pi_1 = \frac{\mu^2}{2\lambda^2+2\lambda\mu+\mu^2} = \frac{\mu^2}{\lambda^2+(\lambda+\mu)^2}$$

$$\Pi_2 = \frac{2\lambda}{\mu}\Pi_1 = \frac{2\lambda}{\cancel{\mu}} \cdot \frac{\mu^{\cancel{2}}}{2\lambda^2+2\lambda\mu+\mu^2} = \frac{2\lambda\mu}{2\lambda^2+2\lambda\mu+\mu^2} = \frac{2\lambda\mu}{\lambda^2+(\lambda+\mu)^2}$$

$$A = \Pi_1+\Pi_2 = \frac{\mu^2+2\lambda\mu}{\lambda^2+(\lambda+\mu)^2} = 1-\frac{2\lambda^2}{\lambda^2+(\lambda+\mu)^2} = 1-\frac{\frac{2}{MTR^2}}{\frac{1}{MTR^2}+\left(\frac{1}{MTR}+\frac{1}{MTBF}\right)^2}$$

Dacă $\lambda = 1/an$ și $\mu = 10/an$: $A = 1-\frac{2\cdot 1^2}{1^2+11^2} = 1-\frac{2}{122} = 0{,}98 = 98\%$