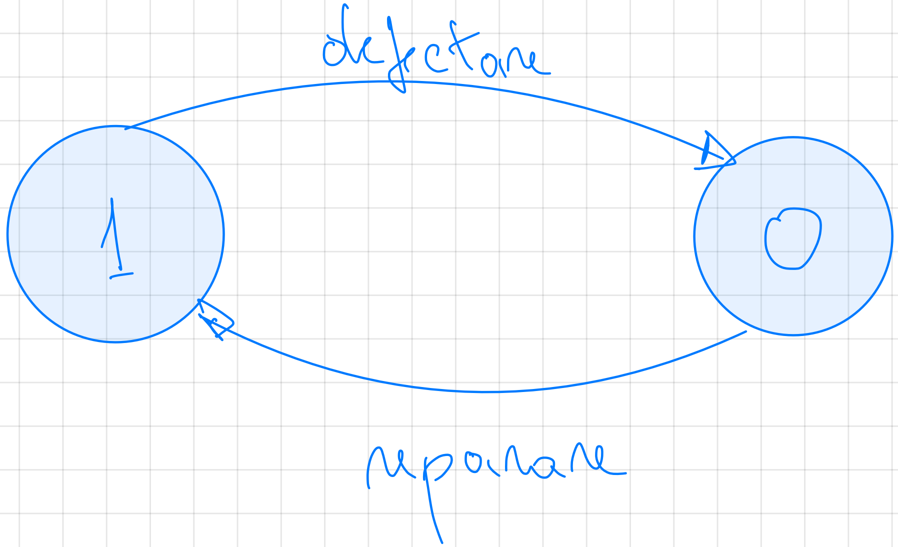
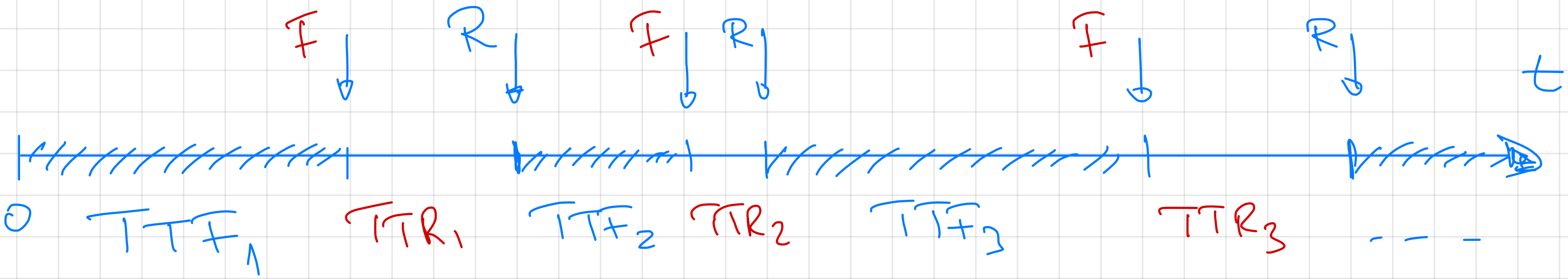
Fiabilitate și toleranță la defecte

$R(t)$ - fiabilitatea

$$R(t) = P(T > t \mid OK @ t=0)$$





$$MTBF = \sum_i \frac{TTF_i}{n}$$

$$MTR = \sum_i \frac{TTR_i}{n}$$

Disponibilitate – $A(t)$

$$A = \frac{\sum_i TTF_i}{\sum_i TTF_i + \sum_i TTR_i} = \frac{MTBF}{MTBF + MTR}$$

Availability (%)	Downtime / year	Downtime / month	Downtime / week
90% ("one nine")	36,5 days	72 h	16,8 h
99% ("2 nines")	3,65 days	7,2 h	1,68 h
99,9% ("3 nines")	8,76 h	43,2 min	10,1 min
99,99% ("4 nines")	52,56 min	4,32 min	1,01 min
99,999%	5,25 min	25,9 s	6,05 s
99,9999%	31,5 s	2,59 s	0,605 s

Probability theory 101

$$0 \leq P(A) \leq 1$$

$$P(\bar{A}) = 1 - P(A)$$

$P(A|B)$ – Prob A cond de B

$$P(A \cdot B) = P(A|B) \cdot P(B)$$

Dacă A și B sunt independente

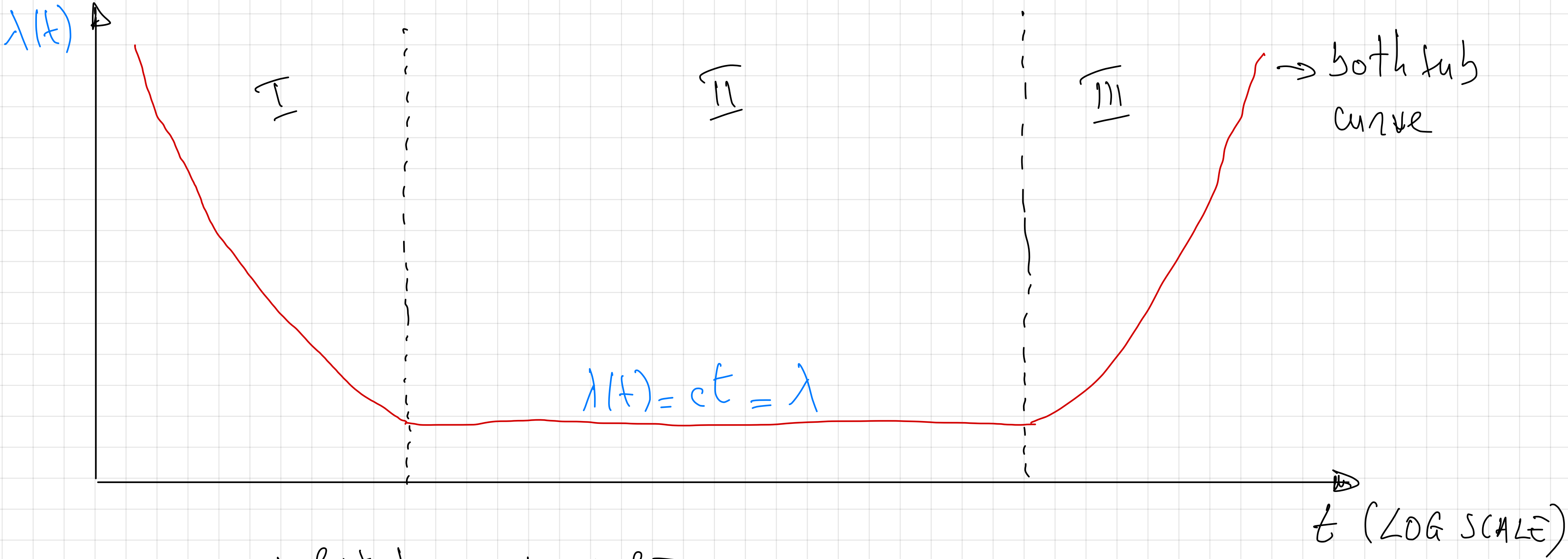
$$P(A \cdot B) = P(A) \cdot P(B)$$

$$P(A+B) = P(A) + P(B) - P(A \cdot B)$$

Dacă A și B sunt mutual exclusive $\Rightarrow P(A \cdot B) = P(B \cdot A) = 0$

$$P(A+B) = P(A) + P(B)$$

Failure Rate → $\lambda(t)$ – intensitatea defecțiunilor

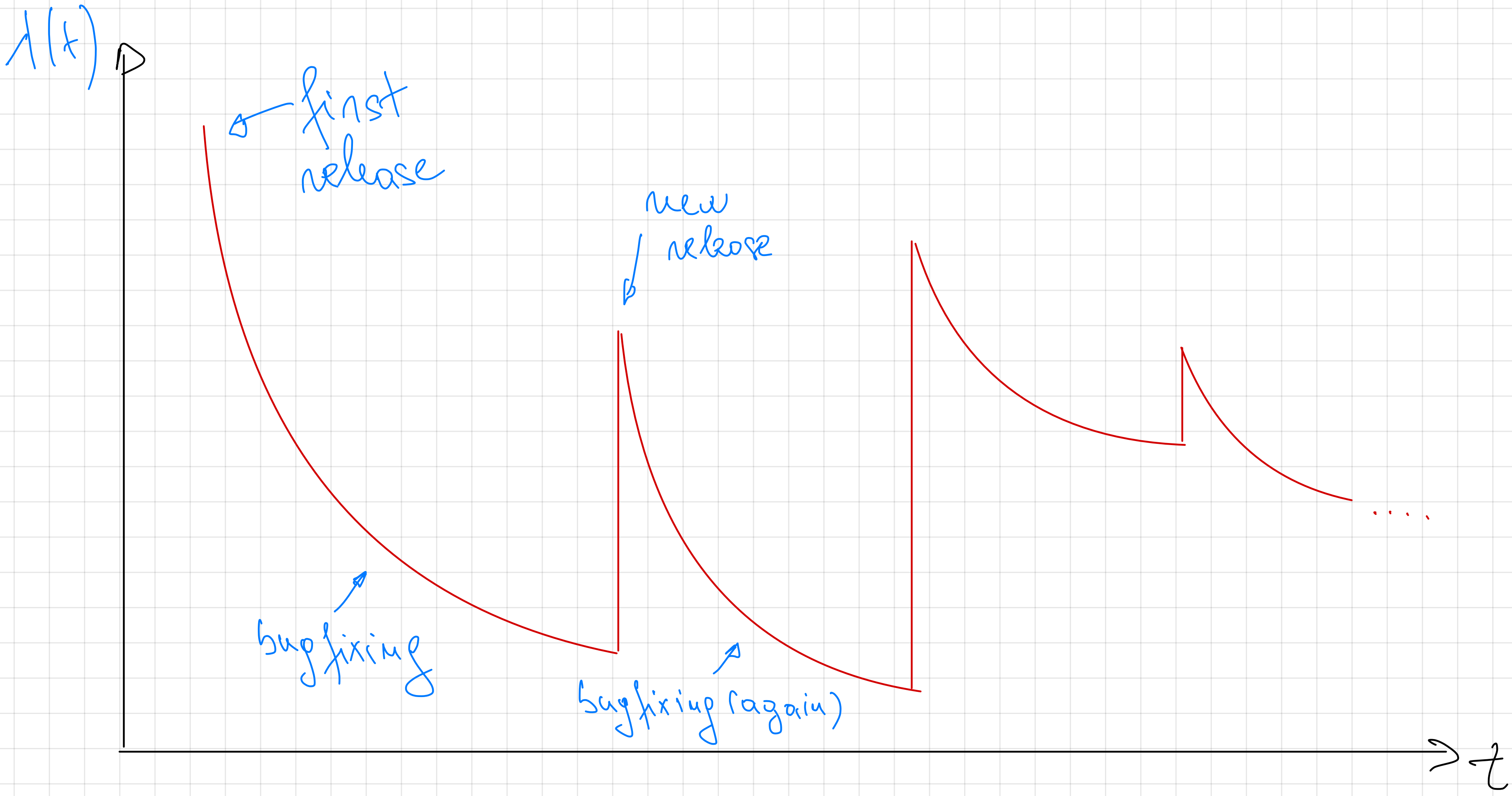


I – mortalitate infantilă

II – viața utilă

III – îmbătrânire

Failure rate - software



Câteva definiții noi:

$f(t)$ – funcție densitate probabilitate (pdf)

$F(t)$ – funcție cumulativă distribuție probabilitate

$$\boxed{f(t) = \frac{dF(t)}{dt}} \qquad \boxed{F(t) = \int_0^t f(\tau)\,d\tau}$$

$$R(t) = 1 - F(t)$$

$$\lambda(t) = \frac{f(t)}{1-F(t)} \Rightarrow \boxed{\lambda(t) = \frac{f(t)}{R(t)}}$$

$$f(t) = \frac{dF(t)}{dt} = \frac{d(1-R(t))}{dt} = -\frac{dR(t)}{dt}$$

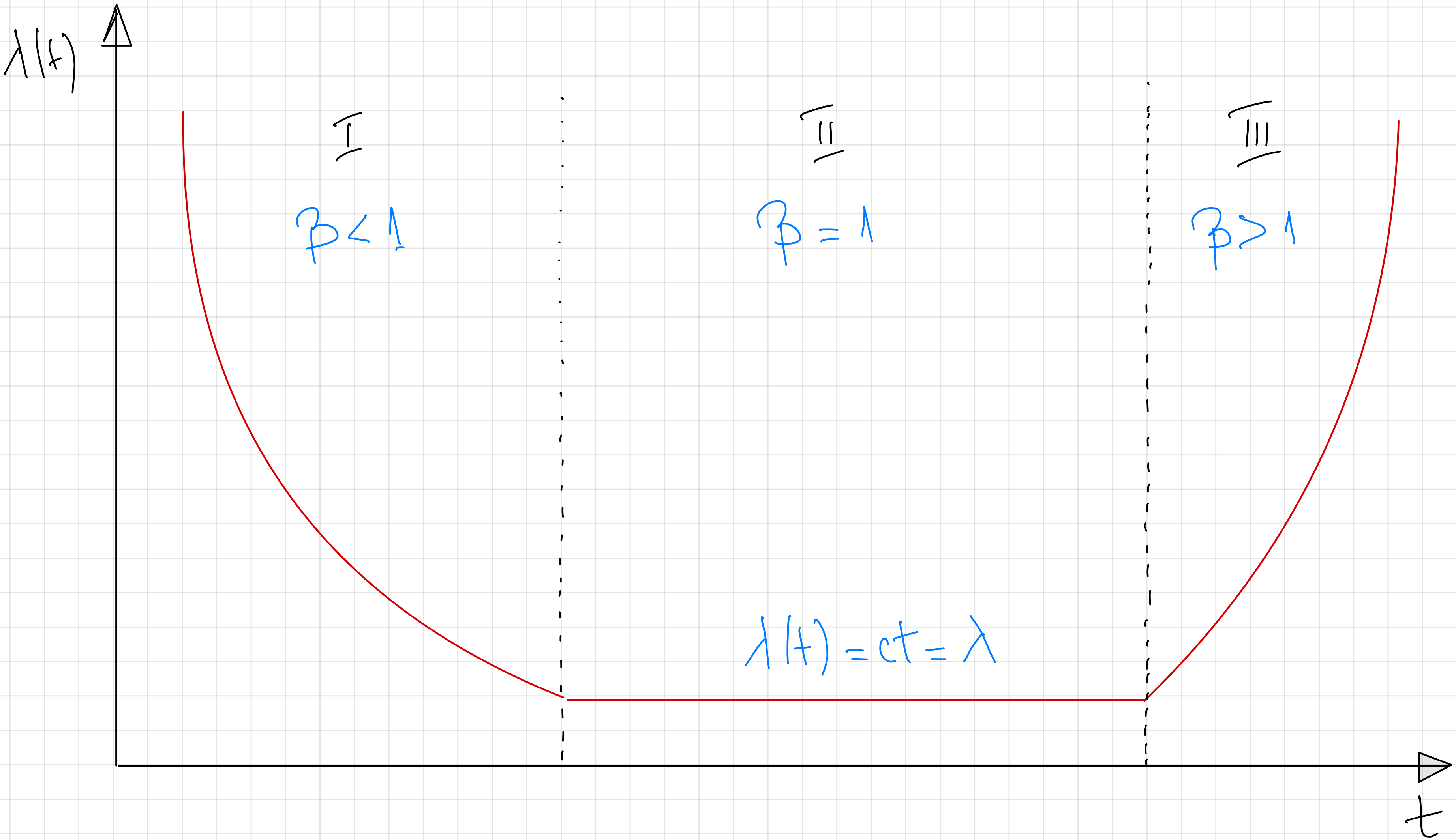
$$\lambda(t) = \frac{-\frac{dR(t)}{dt}}{R(t)} = -\frac{1}{R(t)}\frac{dR(t)}{dt}$$

Aproximare matematică a $\lambda(t)$ – folosim distribuția Weibull

$$f(t) = \lambda\beta t^{\beta-1} e^{-\lambda t^{\beta}}$$

$$\lambda(t) = \lambda\beta t^{\beta-1}$$

- dacă $\beta < 1 \Rightarrow \lambda(t) \searrow$ (zona I pe grafic)
- dacă $\beta = 1 \Rightarrow \lambda(t) = ct$ (zona II pe grafic)
- dacă $\beta > 1 \Rightarrow \lambda(t) \nearrow$ (zona III pe grafic)



1. Cazul în care $\lambda(t) = ct = \lambda$ (perioada de viață utilă a produsului)

$$\lambda(t) = \lambda = -\frac{1}{R(t)}\frac{dR(t)}{dt} \Leftrightarrow \lambda\, dt = -\frac{1}{R(t)}\, dR(t) \Rightarrow$$

$$\Rightarrow \int \lambda\, dt = -\int \frac{1}{R(t)}\, dR(t) \Rightarrow \lambda t + C_1 = -\ln R(t) + C_2 \Rightarrow$$

$$\Rightarrow \ln R(t) = -\lambda t + C \Rightarrow R(t) = e^{-\lambda t + C} = K \cdot e^{-\lambda t}$$

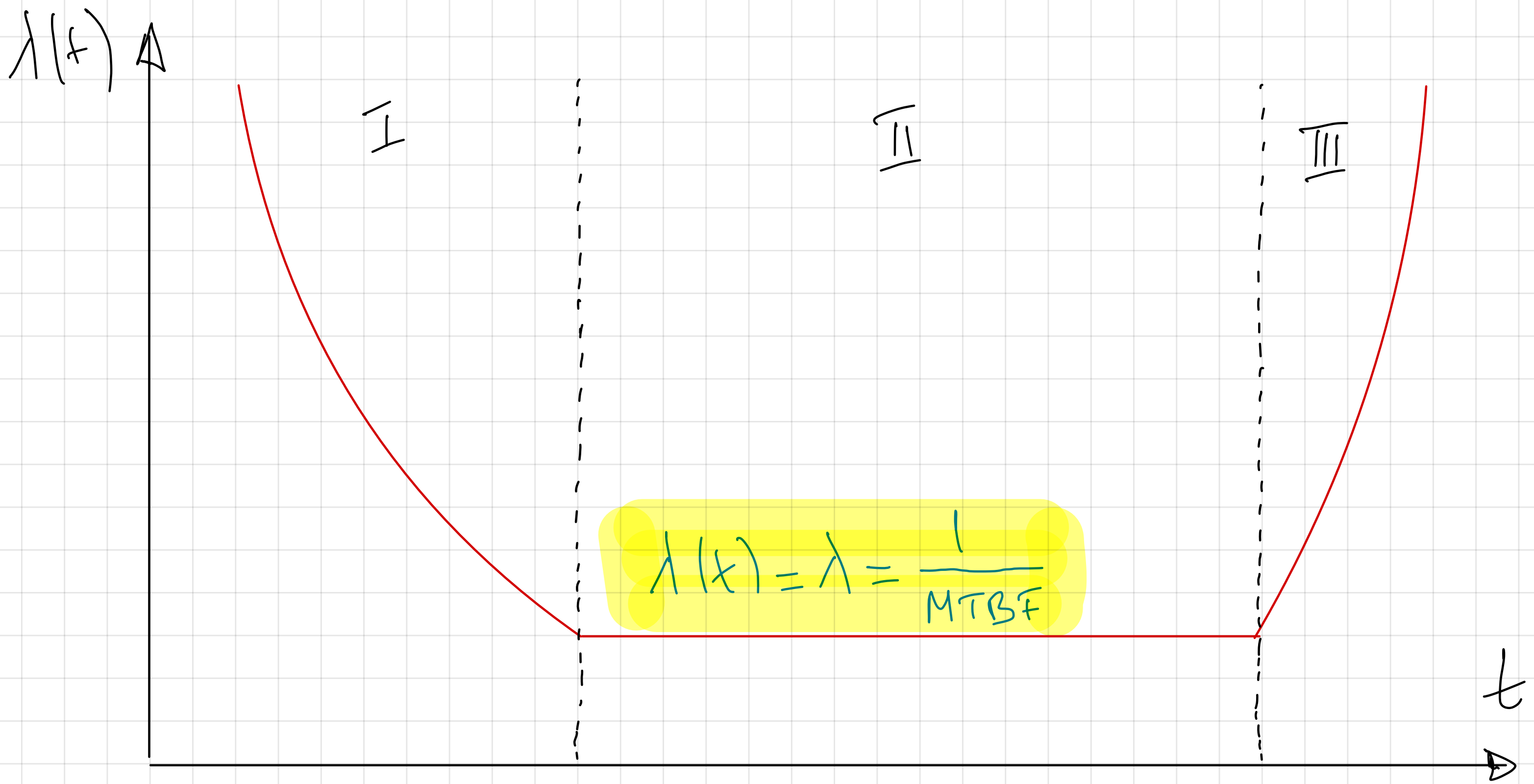
Dacă $t = 0 \Rightarrow R(0) = K$ – putem să presupunem că sistemul pornește la momentul zero de timp în stare de funcționare, deci $R(0) = 1$ (fiabilitate 100%) – alegem $K = 1 \Rightarrow$

$$\boxed{R(t) = e^{-\lambda t}}$$

$$MTBF = \int_0^\infty R(t)\,dt = \int_0^\infty e^{-\lambda t}\,dt = -\frac{1}{\lambda} e^{-\lambda t}\Big|_0^\infty = -\frac{1}{\lambda}\left(e^{-\lambda\infty} - e^{-\lambda\cdot 0}\right) =$$

$$= -\frac{1}{\lambda}(0 - 1) = \frac{1}{\lambda}$$

Deci, pt $\lambda(t) = ct = \lambda$, $MTBF = \dfrac{1}{\lambda}$



2. Dacă $\lambda(t) \neq ct$ (pt. zonele mortalitate infantilă și îmbătrânire)

$$f(t) = \lambda\beta t^{\beta-1} e^{-\lambda t^{\beta}}$$

$$F(t) = \int_0^t f(z)dz = \int_0^t \lambda\beta z^{\beta-1} e^{-\lambda z^{\beta}} dz = \ldots = 1 - e^{-\lambda t^{\beta}}$$

dar $R(t) = 1 - F(t) \Rightarrow R(t) = e^{-\lambda t^{\beta}}$

$$MTBF = \int_0^\infty R(t)dt = \int_0^\infty e^{-\lambda t^{\beta}} dt = \frac{\Gamma(\beta^{-1})}{\beta\lambda^{\beta^{-1}}},$$ unde $\Gamma(x)$ este funcția gamma, definită prin: $\Gamma(x) = \int_0^\infty e^{-y} y^{x-1} dy$ și este extensia pt. numere reale a funcției factoriale $(n!)$

$$\Gamma(x) = (-1+x)\,\Gamma(-1+x)$$

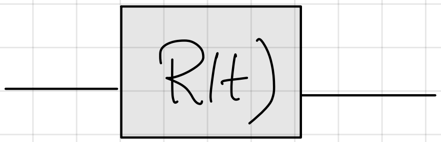
$$\Gamma(x) = \frac{\Gamma(x+1)}{x}$$

$$\Gamma(0) = \Gamma(1) = 1$$

Estimarea fiabilității

Folosim diagrame pentru a modela și estima fiabilitatea

Modularizare: orice sistem, oricât de complex, poate fi modelat fiabilistic ca un modul de fiabilitate dată $R(t)$:

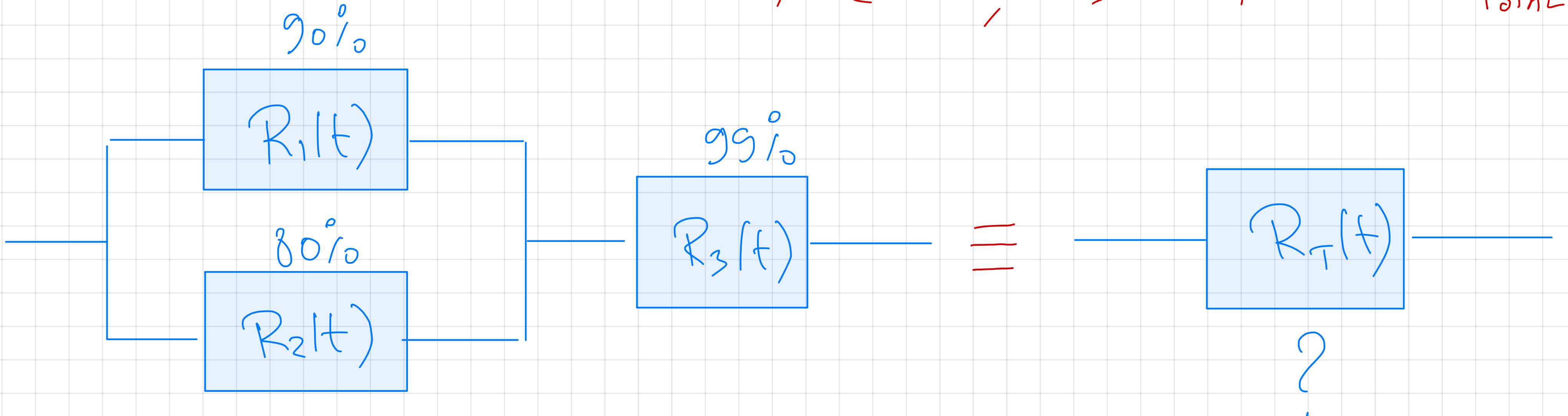


→ Poate fi orice dispozitiv (procesor, calculator, telefon etc) sau orice secvență de cod (bibliotecă, funcție, clasă etc.)

<u>De ex.</u>: Două procesoare ce accesează aceeași memorie.

Primul procesor are o fiabilitate $R_1(t)$, al doilea $R_2(t)$ și memoria $R_3(t)$

Dacă $R_1 = 90\%$, $R_2 = 80\%$ și $R_3 = 99\%$, cât este R_{TOTAL}?

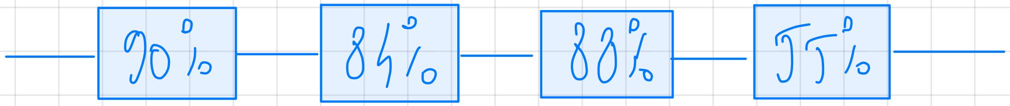


Structuri serie – n module în serie

$$R_s(t) = R_1(t)\cdot R_2(t)\cdot \ldots \cdot R_n(t) = \prod_{i=1}^{n} R_i(t)$$

$$R_s(t) = \prod_{i=1}^{n} R_i(t)$$

De ex.:



$$R_s = 0{,}9\cdot 0{,}84\cdot 0{,}88\cdot 0{,}55 = 0{,}3659 \simeq 0{,}37 = 37\%$$

Dacă $R_i(t) = e^{-\lambda_i t}$

$$R_s(t) = \prod_{i=1}^{n} e^{-\lambda_i t} = e^{-\sum_{i=1}^{n}\lambda_i t}$$

$$R_s(t) = e^{-\sum_{i=1}^{n}\lambda_i t} \qquad \lambda_s = \sum_{i=1}^{n}\lambda_i$$

Definim $Q_i(t)$ → inversul fiabilității

$$Q_i(t) = 1 - R_i(t)$$

$$R_s(t) = \prod_{i=1}^{n} R_i(t)$$

$$Q_s(t) = 1 - R_s(t) = 1 - \prod_{i=1}^{n} (1 - Q_i(t))^n$$

R_i este mare $\geq 90\%$ ⇒ $Q_i \leq 10\%$ $\quad Q_i Q_j \simeq 0$

$$Q_s(t) = 1 - \left(1 - \sum_{i=1}^{n} Q_i + \sum_{\substack{i=1 \\ j=1 \\ i \neq j}}^{n} \underbrace{Q_i Q_j}_{\simeq 0} \ldots\right) \simeq 1 - \left(1 - \sum_{i=1}^{n} Q_i\right) \Rightarrow$$

$$Q_s(t) = \sum_{i=1}^{n} Q_i(t)$$

$$MTBF_i = \int_0^\infty R_i(t)\,dt = \int_0^\infty e^{-\lambda_i t}\,dt = \frac{1}{\lambda_i}$$

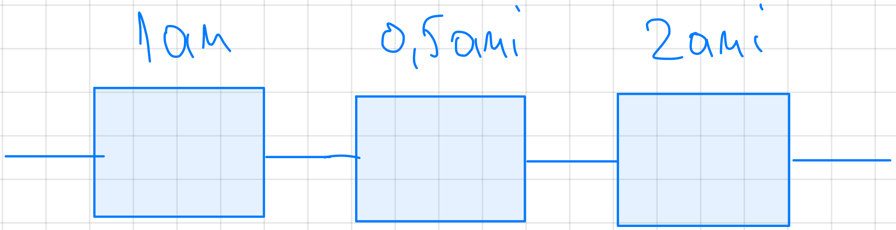
$$MTBF_S = \int_0^\infty R_S(t)\,dt = \int_0^\infty e^{-\sum_{i=1}^{n}\lambda_i t}\,dt = \int_0^\infty e^{-\lambda_s t}\,dt =$$

$$= -\frac{1}{\lambda_s} e^{-\lambda_s t}\Big|_0^\infty = -\frac{1}{\lambda_s}(0-1) = \frac{1}{\lambda_s}$$

$$MTBF_S = \frac{1}{\lambda_S}$$

Dar $\lambda_i = \frac{1}{MTBF_i} \Rightarrow$

$$MTBF_S = \frac{1}{\lambda_1+\lambda_2+\dots\lambda_n} = \frac{1}{\frac{1}{MTBF_1}+\frac{1}{MTBF_2}+\dots\frac{1}{MTBF_n}} = \frac{1}{\sum_{i=1}^{n}\frac{1}{MTBF_i}}$$



$$MTBF_S = \frac{1}{\frac{1}{1}+\frac{1}{0,5}+\frac{1}{2}} = \frac{1}{3+\frac{1}{2}} = \frac{1}{\frac{7}{2}} = \frac{2}{7}\text{ ani}$$

$$MTBF_S = 0,285\text{ ani}$$

Dacă avem n module identice în serie, atunci:

$$R_1(t) = R_2(t) = \dots R_n(t) = R(t) = e^{-\lambda t}$$

$$R_S(t) = \prod_{i=1}^{n} R(t) = \prod_{i=1}^{n} e^{-\lambda t} = e^{-n\lambda t}$$

$$R_S(t) = e^{-n\lambda t} = R^n(t)$$

$$\lambda_S(t) = \sum_{i=1}^{n} \lambda_i(t) = \sum_{i=1}^{n} \lambda = n\lambda$$

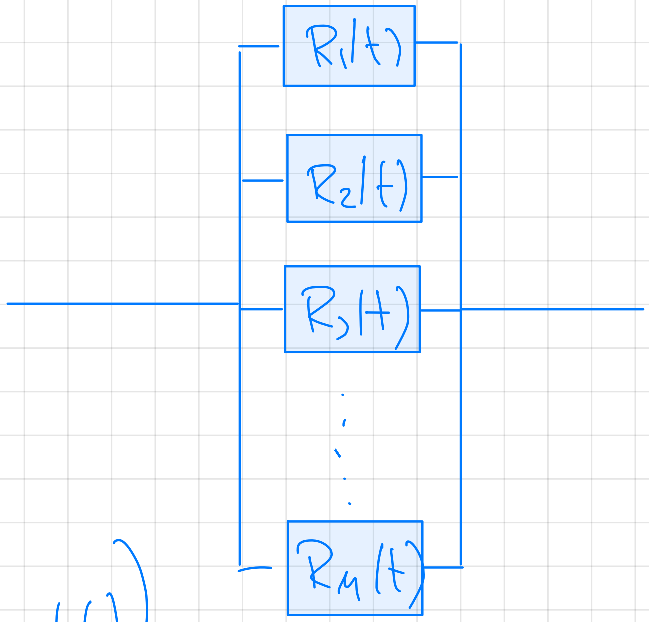
$$\lambda_S = n\lambda$$

$$MTBF_S = \frac{1}{\lambda_S} = \frac{1}{n\lambda} = \frac{MTBF}{n}$$

$$MTBF_S = \frac{MTBF}{n}$$

În general, pt structura serie, R total scade, λ total crește și MTBF total scade.

Structura paralel



$$R_p(t) = ? = 1 - Q_p(t)$$

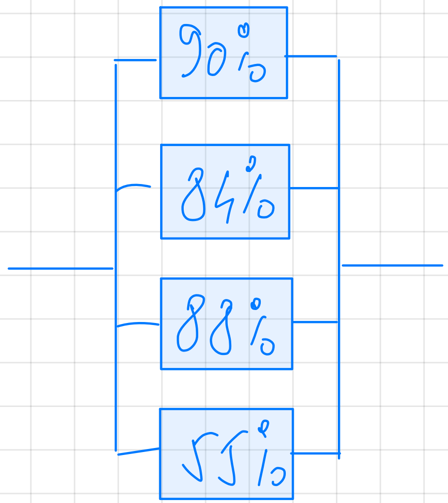
$$Q_p(t) = Q_1(t)\cdot Q_2(t) \dots Q_n(t) =$$

$$= (1-R_1(t))(1-R_2(t)) \dots (1-R_n(t))$$

$$R_p(t) = 1 - (1-R_1(t))(1-R_2(t)) \dots (1-R_n(t))$$

$$R_p(t) = 1 - \prod_{i=1}^{n}(1-R_i(t))$$

<u>De ex</u>: patru module in paralel



$$R_p = 1-(1-0,9)(1-0,84)(1-0,88)(1-0,55) =$$

$$= 1 - 0,1 \cdot 0,16 \cdot 0,12 \cdot 0,45 = 0,9991$$

$$R_p = 99,91\%$$

$$R_i(t) = e^{-\lambda_i t} \Rightarrow R_p(t) = 1 - \prod_{i=1}^{n} \left(1 - e^{-\lambda_i t}\right)$$

$$R_p(t) = 1 - \left(1 - e^{-\lambda_1 t}\right)\left(1 - e^{-\lambda_2 t}\right) \dots \left(1 - e^{-\lambda_n t}\right) \stackrel{?}{=} e^{-\lambda_p t}$$

Presupunem că $R_1(t) = R_2(t) = \dots R_n(t) = e^{-\lambda t}$

$$\boxed{R_p(t) = 1 - \left(1 - e^{-\lambda t}\right)^n}$$

$$\lambda_p(t) = \frac{f_p(t)}{R_p(t)} = \frac{-\frac{dR_p(t)}{dt}}{R_p(t)}$$

$$\frac{dR_p(t)}{dt} = -n\left(1 - e^{-\lambda t}\right)^{n-1}\left(1 - e^{-\lambda t}\right)' = -n\lambda e^{-\lambda t}\left(1 - e^{-\lambda t}\right)^{n-1}$$

$$\boxed{\lambda_p(t) = n\lambda \frac{e^{-\lambda t}\left(1 - e^{-\lambda t}\right)^{n-1}}{1 - \left(1 - e^{-\lambda t}\right)^n}}$$ – nu este constantă!

$$\lambda_{P,\text{steady state}} = \lim_{t\to\infty} \lambda_P(t) = \lim_{t\to\infty} n\lambda \frac{e^{-\lambda t}(1-e^{-\lambda t})^{n-1}}{1-(1-e^{-\lambda t})^n} =$$

$e^{-\lambda t} = x$, dacă $t \to \infty$, atunci $x \to 0$

l'Hospital

$$= \lim_{x\to 0} n\lambda \frac{x(1-x)^{n-1}}{1-(1-x)^n} = n\lambda \lim_{x\to 0} \frac{\left(x(1-x)^{n-1}\right)'}{\left(1-(1-x)^n\right)'} =$$

$$= n\lambda \lim_{x\to 0} \frac{(1-x)^{n-1} - x(n-1)(1-x)^{n-2}}{+n(1-x)^{n-1}} = n\lambda \lim_{x\to 0} \frac{1-x-x(n-1)}{n(1-x)}$$

$$= n\lambda \cdot \frac{1}{n} = \lambda$$

$\boxed{\lambda_P = \lambda}$, la steady-state

$$MTBF_p = \int_0^\infty R_p(t)\,dt = \int_0^\infty \left[1 - \prod_{i=1}^{n} (1 - R_i(t))\right] dt$$

Dacă modulele sunt identice $R(t) = e^{-\lambda t}$

$$MTBF_p = \int_0^\infty \left(1 - (1 - e^{-\lambda t})^n\right) dt$$

$$(1 - e^{-\lambda t})^n = 1 - n e^{-\lambda t} + C_n^2 e^{-2\lambda t} - C_n^3 e^{-3\lambda t} + \dots + (-1)^n e^{-n\lambda t}$$

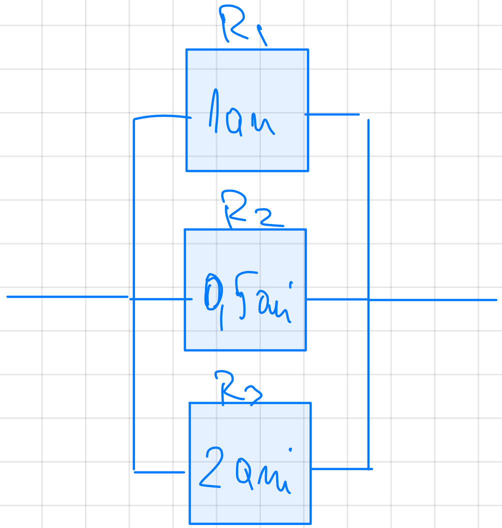
$$MTBF_p = \int_0^\infty \left(n e^{-\lambda t} - \frac{n(n-1)}{2} e^{-2\lambda t} + \dots + (-1)^{n+1} e^{-n\lambda t} \right) dt =$$

$$= n \frac{1}{\lambda} - \frac{n(n-1)}{2} \frac{1}{2\lambda} + \dots + (-1)^{n+1} \frac{1}{n\lambda} = \frac{1}{\lambda} \left(n - \frac{n(n-1)}{4} + \dots (-1)^{n+1} \frac{1}{n} \right)$$

$$= \boxed{\frac{1}{\lambda} \sum_{i=1}^{n} C_n^i \frac{1}{i} (-1)^{i+1}}$$

În general, R total crește, λ total rămâne constant și MTBF total crește

De ex.: trei module in paralel, $MTBF_1 = 1$ an, $MTBF_2 = 0,5$ ani, $MTBF_3 = 2$ ani



$$MTBF_P = \int_0^\infty R_P(t)\,dt = \int_0^\infty \left(1 - (1-R_1)(1-R_2)(1-R_3)\right) dt$$

$$(1-R_1)(1-R_2)(1-R_3) = (1 - R_2 - R_1 + R_1R_2)(1-R_3) =$$

$$= 1 - R_1 - R_2 + R_1R_2 - R_3 + R_2R_3 + R_1R_3 - R_1R_2R_3$$

$$\int_0^\infty \left(R_1 + R_2 + R_3 - R_1R_2 - R_2R_3 - R_1R_3 + R_1R_2R_3\right) dt$$

$$\int_0^\infty \left(e^{-\lambda_1 t} + e^{-\lambda_2 t} + e^{-\lambda_3 t} - e^{-(\lambda_1+\lambda_2)t} - e^{-(\lambda_2+\lambda_3)t} - e^{-(\lambda_1+\lambda_3)t} + e^{-(\lambda_1+\lambda_2+\lambda_3)t}\right) dt$$

$$= \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} - \frac{1}{\lambda_1+\lambda_2} - \frac{1}{\lambda_2+\lambda_3} - \frac{1}{\lambda_1+\lambda_3} + \frac{1}{\lambda_1+\lambda_2+\lambda_3} =$$

$$= MTBF_1 + MTBF_2 + MTBF_3 - \frac{1}{\frac{1}{MTBF_1} + \frac{1}{MTBF_2}} - \ldots = 1 + 0,5 + 2 - \frac{1}{1+2} - \frac{1}{2+\frac{1}{2}} - \frac{1}{1+\frac{1}{2}}$$

$$+ \frac{1}{1+2+\frac{1}{2}} = 3,5 - \frac{1}{3} - \frac{2}{5} - \frac{2}{3} + \frac{2}{7} = 2,385 \text{ ani}$$

Concluzii:

	n module identice în Serie	n module identice în Paralel
R_{TOTAL}	$R_s(t) = R^n(t) = e^{-n\lambda t}$	$R_p(t) = 1 - (1 - R(t))^n = 1 - (1 - e^{-\lambda t})^n$
λ_{TOTAL}	$\lambda_S = n\lambda$	$\lambda_p(t) = n\lambda \frac{e^{-\lambda t}(1 - e^{-\lambda t})^{n-1}}{1 - (1 - e^{-\lambda t})^n}$, $\lambda_{P,stabil} = \lambda$
$MTBF_{TOTAL}$	$MTBF_S = \frac{MTBF}{n}$	$MTBF_p = MTBF \cdot \sum_{i=1}^{n} (-1)^{i+1} \cdot C_n^i \frac{1}{i}$