# The *m*-Distribution—A General Formula of Intensity Distribution of Rapid Fading

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**Abstract**—This paper summarizes the principal results of a series of statistical studies in the last seven years on the intensity distributions due to rapid fading.

The method of derivation and the principal characteristics of the *m*-distribution, originally found in our h.f. experiments and described by the author, are outlined. Its applicability to both ionospheric and tropospheric modes of propagation is fairly well confirmed by some observations. Its theoretical background is also discussed in detail. A theoretical interpretation of the log-normal distribution is given on the basis of this formula. An extremely simplified method is presented for estimating the improvement available from various systems of diversity reception. The mutual dependences between the *m*-formula and other basic distributions are fully discussed. Some generalized forms of the basic distributions are also investigated in relation to the *m*-formula. Two methods of approximating a given function with the *m*-distribution are shown. The joint distribution of two variables, each of which follows the *m*-distribution, is derived in two different ways. Based on this, some useful associated distributions are also discussed.

#### 1. INTRODUCTION

In recent years, radio engineering requirements have become more stringent and necessitate not only more detailed information on median signal intensity, but also much more exact knowledge on fading statistics in both ionospheric and tropospheric modes of propagation.

Such circumstances have forced a large number of extensive experiments and numerous theoretical investigations to be performed on the intensity distribution of fading under various conditions. In order to describe closely the results of these comprehensive observations, diverse forms of the distribution have been presented up to now. Among them, the following three may be regarded, in view of practical uses, as the representatives:

One is the Rayleigh distribution

$$p(R) = \frac{2R}{\Omega} e^{-(R^2/\Omega)}, \qquad (1)$$

where  $\Omega = \overline{R^2}$ , time average of  $R^2$ . This was, as is well known, derived theoretically by Lord RAYLEIGH (1880). Since PAWSEY'S (1935) experimental verification in h.f., many investigators have also confirmed the applicability of this form to fading in both modes of propagation, under scattering conditions at least.

Another is the log-normal distribution

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$$p(\chi) = \frac{1}{\sqrt{2\pi\sigma_{\chi}}} e^{-(\chi-\bar{\chi})^2/2\sigma_{\chi}^2}, \qquad (2)$$

where  $\chi$  denotes signal intensity in terms of db. This seems to have been first

introduced in fading problems by GROSSKOPF (1953) to describe his extensive observations which were made over a relatively long period. Its theoretical background is approximately explained in a general manner, as is well known, by the property of the logarithms of positive variates. The author (Japan (NAKAGAMI), 1955) gave it a satisfactory explanation in connection with his *m*-distribution, by taking into account the average-intensity variations which ought to exist for such longer intervals as in GROSSKOPF's observations.

A third is the m-distribution, named and proposed by the author, whose functional form is

$$p(R) = \frac{2m^m R^{2m-1}}{\Gamma(m)\Omega^m} e^{-(m/\Omega)R^2}, \qquad (3)$$

where  $\Omega = \overline{R^2}$ , and *m* is

$$m = {(R^2)^2 \over (R^2 - \overline{R^2})^2} \ge {1 \over 2}, \text{ always},$$
 (4)

that is, the inverse of the normalized variance of  $R^2$ . This formula was deduced by NAKAGAMI (1943) from his large-scale experiments on rapid fading in h.f. longdistance propagation. Some recent observations in h.f. (e.g. WAMBECK and Ross, 1951) seem to well confirm its applicability. The author also found in some recent data that his distribution accounts for the observations at 4000 mc (NAKAGAMI, 1951) better than the other distributions, and that its applicability can be extended without difficulty to a wider range from 200 mc to 4000 mc (NAKAGAMI and FUJIMURA, 1953). Further, some more recent observations (e.g. MATSUO and IKEDA, 1953; BULLINGTON, INKSTER and DURKEE, 1955, see Figs. 11 and 12) seem to support strongly this formula for tropospheric fading under various conditions. It is of interest to remark that this formula can be regarded to be, in a sense, a generalized form of the Rayleigh distribution, for it includes the latter as its special case of *m* equal to unity.

On the theoretical side, on the other hand, in addition to the Rayleigh distribution, the following two compact forms of distribution,

$$p(R) = \frac{2R}{\sigma} e^{-(R^2 + R_0^2)/\sigma} I_0\left(\frac{2RR_0}{\sigma}\right)$$
(5)

and

$$p(R) = \frac{2R}{\sqrt{\alpha\beta}} e^{-(R^2/2)(1/\alpha + 1/\beta)} I_0 \left[\frac{R^2}{2} \left(\frac{1}{\beta} - \frac{1}{\alpha}\right)\right], \qquad (6)$$

were presented by NAKAGAMI (1940a) and by NAKAGAMI and SASAKI (1942a), respectively. These were found, in 1939 and in 1941 respectively, in a series of their theoretical investigations on fading as particular solutions of the so-called problem of random interference, which is reasonably considered to be the main cause of rapid fading. The former, named "*n*-distribution", is frequently used in radio engineering. The latter, named "*q*-distribution", also appears in communication problems. More recently, we (NAKAGAMI, WADA and FUJIMURA, 1953) proved that the *m*-distribution is a more general solution with good approximation to the random vector problem. At the same time it was also shown that the m-distribution includes in a particular manner the two distributions stated above. Also, the mutual dependences among their parameters were fully investigated (NAKAGAMI, WADA and FUJIMURA, 1953), when the m-distribution and the other distributions were mutually transformed.

The foregoing descriptions suggest that the *m*-distribution is well qualified as a representative distribution at least at the present status. In further developing the foregoing theories of the *m*-distribution, we (NAKAGAMI and NISHIO, 1953, 1954a, 1955) have also established in two different manners the joint distribution function of two variables following the *m*-distribution law. Based upon this formula, a new unified theory of diversity effects (NAKAGAMI and NISHIO, 1953, 1955) and a more general, but the simplest, method (NAKAGAMI and YOKOI, 1953; NAKAGAMI, 1953) of evaluating the improvements available from various systems of diversity reception have been proposed.

Further, an entirely new method (e.g. NAKAGAMI, TANAKA and KANEKO, 1954) of observing long term distributions was found, based on the foregoing theories, in order to avoid various inconveniences in the usual methods of observation and in handling quantities of data. Using this apparatus, we are now obtaining much useful information on fading (e.g. NAKAGAMI and TANAKA, 1956).

Most of the original papers of the author are written in Japanese, and it has long been his regret that they appear to be little known abroad. Dr. Hoffman's kind invitation to write a paper based on our summary report (NAKAGAMI, TANAKA and KANEHISA, 1957) for these proceedings was therefore well received. The major portion of the present paper is a condensation of that report together with some unpublished results, but because of space limitations, the graphs and references are cut to the minimum. For more detailed information on our results, readers are referred to the original papers, which are completely listed in the bibliography at the end of this paper.

## 2. OUTLINE OF THE ORIGINAL DERIVATION OF THE *m*-DISTRIBUTION AND SOME OF ITS BASIC PROPERTIES

## 2.1 Outline of the Original Derivation

(a) Time interval of observation. In order to observe rapid fading alone, i.e. to remove the effect of slow fading, the length of observation interval should be carefully chosen, because the effect of slow fading will be more predominant for too long time intervals, while the statistical meaning of the observed distribution becomes ambiguous for too short time intervals. Therefore, there must exist an optimum length of observation interval. This length, of course, depends on various factors such as frequency, propagation path, the time of the day, etc. After careful preliminary tests for h.f. long-distance propagation, the interval was determined as about three to seven minutes in our experiments.

(b) Apparatus and observed waves. A vertical antenna, about 1.5 m long, was used, the output of which, after amplification, logarithmic compression and envelope detection, was applied to the deflection plates of a CR tube. The

movement of the spot on the fluorescent screen follows exactly the signal variation, which is recorded on a photographic plate placed in front of the screen. The required distribution was determined by measuring the emulsion density of the plate after developing. Special care was taken to make the measurements exact. The overall time constant of the apparatus was 2 milliseconds (maximum).

The signals observed and the number of plates used in deducing the experimental formula are shown in Table 2.1.

Station	Distance (km)	Call sign	Frequency	Date	No. of plates	No. of plates $m < 0.5$
Changehun	1,500	JMP2	10,065 kc	Apr. 1941	34	0
Palau	3,200	JRAK	11,740 kc	Oct. 1941	15	0
a <b>1</b> 7 1		KNY	19,080 kc	Nov. 1940– Oct. 1941	67	1
San Francisco	8,240	KGEI	9,670 kc	Oct. 1941	17	0
		KWU	15,355 kc	Oct. 1941	14	0
Berlin	8,900	DFZ	20,020 kc	Apr. 1941	19	0
Taipei	2,200	JIB	10,535 kc	Oct. 1940– Dec. 1941	265	0

Table 2.1

(c) Example. An example of the records is shown in the photograph, in which ordinate and abscissa indicate each of two spaced-antenna outputs in decibels respectively. These records were used primarily to estimate the amount of diversity effects which can be expressed in terms of the correlation coefficient between the two outputs, and secondarily to reduce the functional form of intensity distributions. This method is, incidentally, similar to that of G. R. SUGAR (1954).

(d) Derivation of the distribution function (7). By inspection, the functional form of the measured distributions had been inferred, at least approximately, to take the form of (7). To check this, they were first plotted on a system of log-log co-ordinates, whose ordinate and abscissa were so chosen that (7) might be represented as a group of straight lines having slopes proportional to the values of m. Almost all the distributions measured are well represented as straight lines on this system. Some of the representative distributions are illustrated in Fig. 2.1.

From this fact, their functional forms were determined as

$$p'(\chi) = \exp\left\{m\left(1 + \frac{2\chi}{M} - e^{2\chi/M}\right)\right\},\qquad(7)$$

where  $\chi$  is the signal intensity in decibels and  $M = 20 \log_{10} e = 8.686$ .

Upon normalization of (7), we obtain the distribution function of dbintensity  $\chi$ 

$$p(\chi) = \frac{2m^m}{M\Gamma(m)} \exp\left\{m\left(\frac{2\chi}{M} - e^{2\chi/M}\right)\right\} \equiv \mathscr{M}_{\chi}(\chi, m, 0).$$
(8)



(See text for explanation)

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X(db) Fig. 2.1 Also, from Table 2.1, we observe that the following holds with only one exception,

$$m \geq \frac{1}{2}.$$
 (9)

Using the transformations  $e^{\chi/M} = X = \frac{R}{\sqrt{\Omega}}$ , where  $\Omega = \overline{R^2}$ , the time average of the square of intensity R, we finally arrived at the distribution (NAKAGAMI, 1943)

$$p(X) = \frac{2m^m}{\Gamma(m)} X^{2m-1} e^{-mX^2} = \mathcal{M}(X, m, 1),$$
(10)

or

$$p(R) = \frac{2m^m R^{2m-1}}{\Gamma(m)\Omega^m} e^{-(m/\Omega)R^2} \equiv \mathscr{M}(R, m, \Omega).$$
(11)



This formula, defining the "*m*-distribution", includes both the Rayleigh distribution and the one-sided Gaussian distribution as special cases for m = 1 and  $m = \frac{1}{2}$ , respectively.

p(X) and  $p(\chi)/p(0)$  are illustrated graphically in Figs. 2.2, and 2.3, respectively. (e) A remark on the *m*-distribution formula. This distribution is apt to be confused with the  $\chi^2$  and the  $\Gamma$ -distribution from their functional similarity, but there exists a somewhat essential difference in the admissible range of values of the parameter, i.e. in the latter two the parameter is usually assumed as a positive integer and a positive number, respectively, while in the former we may assign any positive number not less than  $\frac{1}{2}$ , as is shown in (9). This significant difference will serve not only better to understand the *m*-distribution from the theoretical viewpoint, but also to distinguish this formula from the other distributions with similar functional forms.

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2.2 Some Properties of the m-Distribution

In the following, some of the prominent features of the m-distribution, which are basic and useful in its applications, will be presented without proof.

(a) Basic properties of  $\mathscr{M}_{\chi}(\chi, m, 0)$ . As readily seen from (8),  $\mathscr{M}_{\chi}(\chi, m, 0)$  has the maximum value

$$p(0) = \frac{2m^m}{M\Gamma(m)e^m} \simeq \frac{1}{M} \sqrt{\frac{2m}{\pi}} \qquad (m \text{ large}), \tag{12}$$

at  $\chi = 0$  or  $R = \sqrt{\Omega}$ . This relation is of practical value. For instance, if we apply this to an observed db-intensity distribution, the effective value of the linear intensity can be found at a glance. When  $\chi \leq M$  in (8),  $\mathscr{M}_{\chi}(\chi, m, 0)$  approaches to the form of log-normal distribution

$$p(\chi) \simeq \frac{1}{M} \sqrt{\frac{2m}{\pi}} e^{-2m(\chi/M)^2}.$$
 (13)

Further,  $\mathcal{M}_{\chi}(\chi, m, 0)$  can be generalized in a form

$$p(\tau) = \frac{2m^m}{M\Gamma(m)} \exp\left[m\left\{\frac{2(\tau-\tau_0)}{M} - e^{2(\tau-\tau_0)/M}\right\}\right] \equiv \mathcal{M}_{\tau}(\tau, m, \tau_0), \quad (14)$$

where  $\tau$  and  $\tau_0$  are db-intensities of R and  $\sqrt{\Omega}$  above unit intensity, respectively.

Graphs of the cumulative distribution defined by

$$M(\chi, m) = \int_{-\infty}^{\chi} \mathcal{M}_{\chi}(\chi, m, 0) \, \mathrm{d}\chi, \qquad (15)$$

are illustrated in two different systems of co-ordinates such as Figs. 2.4 and 2.5. These are of great use in practical applications.



Finally, the characteristic function becomes

$$\phi(z) = \int_{-\infty}^{\infty} \mathscr{M}_{\chi}(\chi, m, 0) e^{-z\chi} d\chi = \frac{\Gamma\left(m - \frac{M}{2}z\right)}{\Gamma(m)} m^{(M/2)z}.$$
 (16)

(b) Moments and variances. First, the moments and the variances will be listed below:

$$\overline{R^{\nu}} = \frac{\Gamma\left(m + \frac{\nu}{2}\right)}{\Gamma(m)} \left(\frac{\Omega}{m}\right)^{\nu/2}, \qquad \overline{R^{2n}} = \left(\frac{\Omega}{m}\right)^n (m + \overline{n-1})(m + \overline{n-2}) \dots m, \qquad (17)$$

$$V(R^2) = \frac{\Omega^2}{m}, \qquad V(R) = \Omega \left[ 1 - \left\{ \frac{\Gamma(m + \frac{1}{2})}{\sqrt{m}\Gamma(m)} \right\}^2 \right] \simeq \frac{\Omega}{5m}, \qquad (18)$$

where v and n are a positive number and a positive integer, respectively. The same notations will be used throughout this paper, unless the contrary is stated.

Next, the moments and the variances of db-intensity are shown in the following:

$$\overline{\chi} = \frac{M}{2} \{ \psi(m) - \log_{e} m \}, \qquad \overline{\chi}^{2} = \left(\frac{M}{2}\right)^{2} [\{ \psi(m) - \log_{e} m \}^{2} + \psi'(m)], \\ \overline{\chi}^{3} = \left(\frac{M}{2}\right)^{3} [\{ \psi(m) - \log_{e} m \}^{3} + 3\psi'(m) \{ \psi(m) - \log_{e} m \} + \psi''(m)],$$
(19)

$$\overline{(\chi-\bar{\chi})^2} = \left(\frac{M}{2}\right)^2 \psi'(m), \qquad \overline{(\chi-\bar{\chi})^3} = \left(\frac{M}{2}\right)^3 \psi''(m), \qquad (20)$$

where  $\psi(\chi)$ ,  $\psi'(\chi)$  and  $\psi''(\chi)$  are the Di-gamma, the Tri-gamma and the Tetragamma function, respectively.

(c) The parameter m. In the m-distribution, the parameter m has an important meaning, which will soon be made clear.

Now, returning to (18) we get the expression

$$m = \frac{\Omega^2}{V(R^2)} = \frac{1}{V_N(R^2)},$$
 (21)

where  $V_N(R^2)$  denotes the normalized variance of  $R^2$ . That is, m is the inverse



of the normalized variance of  $R^2$  exactly. This relation suggests the possibility to use *m* as a measure of fading range N(P) defined by  $\chi_2 - \chi_1$ , where

$$P = \int_{-\infty}^{\chi_1} \mathscr{M}_{\chi}(\chi, m, 0) \, \mathrm{d}\chi = \int_{\chi_2}^{\infty} \mathscr{M}_{\chi}(\chi, m, 0) \, \mathrm{d}\chi.$$

This suggestion was justified fairly well by numerical calculations (NAKAGAMI, 1955) as

$$N(P) \simeq 10 \left( \frac{1}{m} + 0.2 \right) \log_{10} \frac{1}{P} + 1.5 \text{ db} \qquad (m < 8).$$
 (22)

From this, it can be seen that N(P) is linearly proportional to 1/m, so it is termed "Fading Figure".

Based upon the above properties of the *m*-distribution, we (NAKAGAMI and FUJIMURA, 1953) have proposed defining the intensity variation in actual fadings by the parameter m instead of by the conventional fading range. The advantages of this definition will be evident from the preceding discussions.

## 2.3. Distribution of Fineness (NAKAGAMI, KANEHISA and OTA, 1955)

In the following, we shall give a short discussion on the distribution of the so-called fineness, which means the rapidity of intensity fluctuations, and is ordinarily expressed by the average number of crossings of a specified intensity level per unit time.

Now, assuming the stationarity of fading, the fineness G(R) at an arbitrary level R can be expressed by

$$G(R) = \int_{-\infty}^{\infty} |R'| \, p(R, R'; t) \, \mathrm{d}R', \qquad (23)$$

where p(R, R'; t) is the joint distribution function of the two random variables R and R', where R' is the time derivative of R. The correlation coefficient between R and R' can be easily shown to vanish in almost all cases that may be encountered in actual fading. This property suggests that R and R' are mutually independent, so that (23) may be written, at least to a good approximation, as

$$G(R) = \int_{-\infty}^{\infty} |R'| \ p(R)p(R') \ \mathrm{d}R' = |\overline{R'}| \times p(R). \tag{24}$$

Here, we could arrive at the conclusion that the fineness takes the same functional form as that of the intensity distribution. This conclusion was proved theoretically in some special cases (e.g. MIDDLETON, 1948) and also suggested by some observations (e.g. MIYA, INOUE and WAKAI, 1953). As to p(R'), it is natural to assume that it will take a Gaussian type in general, at least to a high degree of approximation. Under this assumption G(R) can be easily reduced, in the case of the *m*-distribution, to

$$G(R) = \sqrt{-\frac{2}{\pi}\varphi''(0)} \mathscr{M}(R, m, 0), \qquad (25)$$

 $\varphi(\tau)$  being the auto-correlation function of R.

## 2.4. Integral Expressions of $\mathcal{M}(R, m, \Omega)$

The *m*-distribution is, like many others, often required in the form of a definite integral in its applications. Such an integral representation can be found in various forms. Among these, the two forms, i.e. the Bromwich type of contour integral and the Hankel type of integral, are of much convenience for the present purpose. Therefore, we shall outline these two forms in the following.

(a) The expression by Bromwich's type of contour integral. This expression is readily reduced by the Laplace transform to

$$\mathscr{M}(R, m, \Omega) = \frac{2R}{2\pi j} \int_{c-j\infty}^{c+j\infty} e^{zR^2} \frac{\mathrm{d}z}{\left(\frac{\Omega}{m}z+1\right)^m}.$$
 (26)

The path of integration is the so-called Bromwich contour, i.e. a straight line parallel to the imaginary axis at a distance c from the origin, c being so chosen that all the singularities of the integrand are on the left side of the line.

(b) The expression by Hankel's type of integral. From the vth Hankel transform, the expression

$$\mathscr{M}(R, m, \Omega) = \frac{R^m}{2^{m-1}\Gamma(m)} \int_0^\infty \lambda^m J_{m-1}(\lambda R) e^{-(\Omega/4m)\lambda^2} \,\mathrm{d}\lambda, \qquad (27)$$

can be derived without difficulty, and the transform of zero order also yields another expression

$$\mathscr{M}(R, m, \Omega) = R \int_0^\infty \lambda J_0(\lambda R) L_{m-1}\left(\frac{\Omega}{4m} \lambda^2\right) e^{-(\Omega/4m)\lambda^2} \,\mathrm{d}\lambda, \qquad (28)$$

after some calculations,  $L_{r}(x)$  being the Laguerre function.

These expressions are, of course, equally valid for all values of  $m \ge \frac{1}{2}$ . These expressions will often be used to advantage in the following discussions.

### 3. THEORETICAL BACKGROUND OF THE *m*-DISTRIBUTION

We now turn our attention to the theoretical basis of the m-distribution, and further to the relationships between the m-formula and the other basic distribution forms stated above.

## 3.1. The m-Distribution as a General Approximate Solution of the General Problem of Random Vectors

Before proceeding to the theoretical background of the m-distribution, we now give some brief descriptions on the uses of the Hankel and the Laplace transforms. They are of much expediency in the statistical treatments of such positive variates as the modulus of a vector, and the distance, in a two- or a multi-dimensional space, etc.

We shall show in the following how to use these transforms in the statistics of fading problems.

(a) Uses of the Hankel transform and the Laplace transform in the fading statistics. Now, let  $\xi(x_1, \ldots, x_n)$  be a given positive function of random variables  $x_1, x_2, \ldots, x_n$ ; then the distribution function of R, equal to  $\xi$ , can be formally expressed as

$$p(R) = \overline{\delta(R - \xi)},\tag{29}$$

where  $\delta(x)$  denotes the  $\delta$ -function after Dirac, and the bar means the average with respect to the random variables.

Here if we use the known relation (e.g. WATSON, 1922)

$$\delta(R - \xi) = R^{\nu+1} \int_0^\infty \lambda J_\nu(\lambda R) \frac{J_\nu(\lambda \xi)}{\xi^\nu} d\lambda \qquad (R(\nu) \ge -\frac{1}{2}), \tag{30}$$

(29) becomes

$$p(R) = \frac{R^{\nu+1}}{2^{\nu}\Gamma(\nu+1)} \int_0^\infty \lambda^{\nu+1} J_{\nu}(\lambda R) F_{\nu}(\lambda) \, \mathrm{d}\lambda, \qquad (31)$$

where

$$F_{\nu}(\lambda) = 2^{\nu} \Gamma(\nu + 1) \left[ \frac{J_{\nu}(\lambda\xi)}{\lambda^{\nu} \xi^{\nu}} \right] \equiv \overline{\Lambda_{\nu}(\lambda\xi)}.$$
(32)

This function is of considerable importance in this method, corresponding to the characteristic function in the usual method in statistics, and named "amplitude characteristic function of  $\nu$ th order".  $\Lambda_{\nu}(x)$  is a tabulated function by JAHNKE and EMDE (1943). By the Hankel inversion theorem,  $F_{\nu}(\lambda)$  may be written as

$$F_{\mathbf{v}}(\lambda) = \int_{0}^{\infty} p(R) \Lambda_{\mathbf{v}}(\lambda R) \, \mathrm{d}R. \tag{33}$$

In particular,

$$p(R) = R \int_0^\infty \lambda J_0(\lambda R) F_0(\lambda) \, \mathrm{d}\lambda,$$
  

$$F_0(\lambda) = \int_0^\infty J_0(\lambda R) p(R) \, \mathrm{d}R.$$
(34)

Integrating (31) with respect to R, we are able to establish a general expression of the cumulative distribution in the compact form of a Hankel integral.

In applying the expressions (31) and (32) to individual problems, the order  $\nu$  of the Bessel function involved may be any positive value not less than  $-\frac{1}{2}$ . But it is preferable to choose such that

$$\nu + 1 = \frac{(\overline{\xi^2})^2}{(\overline{\xi^2} - \overline{\xi^2})^2} \equiv m,$$
 (35)

because the calculation of  $F_{\nu}(\lambda)$  becomes simpler, the reason for which will be seen in later calculations.

Further, in some applications of this method, it is worth noticing that if  $\xi$  is an *n*-dimensional vector, then the order  $\nu$  of the Bessel function involved is connected with the number of dimensions *n* by the relation

$$\nu = \frac{n}{2} - 1, \tag{36}$$

as is well known in the theory of the Hankel transforms. Based on this, we could treat any given positive variate from the multi-dimensional viewpoint.

Returning to (29), if we take

$$\delta(R - \xi) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} e^{z(R^2 - \xi^2)} \,\mathrm{d}z,$$

it yields the integral expression

$$p(R) = \frac{2R}{2\pi j} \int_{c-j\infty}^{c+j\infty} e^{zR^2} \phi(z) \, \mathrm{d}z,$$

$$\phi(z) = [\overline{e^{-z\xi^2}}],$$
(37)

where

which is usually called the moment-generating function.

It is of interest to see that  $\phi(z)$  and  $F_{p}(\lambda)$  are generally connected by the following Laplace transform

$$g(s) = \int_0^\infty f(t)e^{-st} \,\mathrm{d}t, \tag{38}$$

where

$$\begin{split} g(s) &= 2^{2(\nu+1)} \Gamma(\nu+1) z^{\nu+1} \phi(z), \qquad s = \frac{1}{4z}, \\ f(t) &= F_{\nu}(\lambda) \lambda^{2\nu}, \qquad t = \lambda^2. \end{split}$$

Furthermore, the applicabilities of the above two methods, of course, can readily be extended to the case of two or more random variables, e.g. in the case of two variables the distribution function is expressed, in the two methods, as

$$p(R_{1}, R_{2}) = \frac{(R_{1}R_{2})^{\nu+1}}{[2^{\nu}\Gamma(\nu+1)]^{2}} \int_{0}^{\infty} \int_{0}^{\infty} \lambda_{1}^{\nu+1} \lambda_{2}^{\nu+1} J_{\nu}(\lambda_{1}R_{1}) J_{\nu}(\lambda_{2}R_{2}) F_{\nu}(\lambda_{1}, \lambda_{2}) \, \mathrm{d}\lambda_{1} \, \mathrm{d}\lambda_{2},$$

$$F_{\nu}(\lambda_{1}, \lambda_{2}) = \overline{\Lambda_{\nu}(\lambda_{1}\xi_{1})\Lambda_{\nu}(\lambda_{2}\xi_{2})},$$
(39)

and

where

$$p(R_1, R_2) = \frac{4R_1R_2}{(2\pi j)^2} \int_{c-j\infty}^{c+j\infty} \int_{c-j\infty}^{c+j\infty} e^{z_1R_1^2 + z_2R_2^2} \phi(z_1, z_2) \, \mathrm{d}z_1 \, \mathrm{d}z_2,$$

$$\phi(z_1, z_2) = \overline{[e^{-(z_1\xi_1^2 + z_2\xi_2^2)]}}$$

$$(40)$$

where

respectively.

The use of the former method is to be found in its origin with the great contributions due to KLUYVER (1906) and PEARSON (1906), and also in our later work (e.g. NAKAGAMI and SASAKI, 1942a) in a somewhat more advanced form. More recently it has been much developed in well-established forms by LORD (1954) and others, especially in our laboratory (e.g. NAKAGAMI, 1954; OTA, 1956; OTA and NAKAGAMI, 1956). The latter method was further developed (e.g. NAKAGAMI, 1940b) in relation to the treatment of transient phenomena in electric circuits.

(b) Derivation of the *m*-distribution (NAKAGAMI, WADA and FUJIMURA, 1953). Regardless of the modes of propagation, i.e. whether ionospheric or tropospheric, it is reasonably supposed that the signal intensity  $\xi$  at an observing point is composed of some component signals  $r_i e^{j\theta_i}$  (i = 1, 2, ..., n) which have traveled on different paths, and whose amplitudes and phases vary according to certain statistical laws. Under these conditions,  $\xi$  may be generally written as

$$\xi = \left| \sum_{i=1}^{n} r_i e^{j\theta i} \right| \equiv |x + jy|.$$
(41)

Now, starting with (32), we get

$$F_{0}(\lambda) = 1 - \bar{\xi}^{2} \frac{\lambda^{2}}{4} + \frac{(\bar{\xi}^{2})^{2}}{(2!)^{2}} \left(\frac{\lambda^{2}}{4}\right)^{2} - \frac{(\bar{\xi}^{2})^{3}}{(3!)^{2}} \left(\frac{\lambda^{2}}{4}\right)^{3} + \ldots,$$

by expanding the Bessel function in a power series. Here if we write

$$\nu + 1 = \frac{(\xi^2)^2}{(\xi^2 - \overline{\xi^2})^2} \equiv m, \qquad \overline{\xi^2} = \Omega,$$
 (42)

and make use of the general properties of moments  $(\overline{\xi^{n+s}})^2 \ge \overline{\xi^{2n}} \times \overline{\xi^{2s}}$  due to LIAPOUNOFF, then after some calculations,  $F_0(\lambda)$  can be approximately reduced to

$$F_0(\lambda) \simeq {}_1F_1\left(m, 1; -\frac{\Omega}{4m}\lambda^2\right) = e^{-(\Omega/4m)\lambda^2} L_{m-1}\left(\frac{\Omega}{4m}\lambda^2\right). \tag{43}$$

Applying this to (34), and using (28) we arrive at

$$p(R) \simeq R \int_0^\infty \lambda J_0(\lambda R) L_{m-1}\left(\frac{\Omega}{4m} \lambda^2\right) e^{-(\Omega/4m)\lambda^2} \,\mathrm{d}\lambda = \mathscr{M}(R, m, \Omega). \tag{44}$$

The approximation in this reduction is sufficiently good enough for engineering problems.

In like manner, using (32), we get

$$F_{\bullet}(\lambda) \simeq e^{-\{\Omega/4(\nu+1)\}\lambda^2},\tag{45}$$

to a high degree of approximation. By evaluating the integral (31), after substituting from (45) into it, we finally get

$$p(R) \simeq R^{\nu+1} \int_0^\infty \lambda^{\nu+1} J_{\nu}(\lambda R) e^{-\{\Omega/4(\nu+1)\}\lambda^2} \,\mathrm{d}\lambda = \mathscr{M}(R, m, \Omega). \tag{46}$$

The foregoing reduction procedure indicates that the *m*-distribution arises not only from random interferences, but also from a more general case of random superposition of random vectorial elements. This theoretical evidence affirms that the *m*-distribution might be a more suitable form for both ionospheric and tropospheric fadings. This was fairly well confirmed by many experiments as stated above.

When the central limit theorem holds, the parameter m takes the form

$$m = \frac{(\sigma + A^2)^2}{(\sigma + A^2)^2 + (B^2 - A^4) + 2A^2B\cos 2(\delta_1 - \delta_2)}$$
(47)

where

$$A^2 = (ar{x})^2 + (ar{y})^2, \qquad B^2 = 4c^2 + (\sigma_x - \sigma_y)^2, \ \sigma_x = \overline{V}(x), \qquad \sigma_y = \overline{V}(y), \qquad c = c(x,y), \qquad \sigma = \sigma_x + \sigma_y, \ an \delta_1 = rac{ar{y}}{ar{x}}, \qquad an 2\delta_2 = rac{2c}{\sigma_x - \sigma_y}.$$

From the well known inequality  $c^2(x, y) \leq \sigma_x \sigma_y$ ,  $B \leq \sigma$  results, and (47) yields the following inequality.

$$\hat{m} \ge \frac{(\sigma + A^2)^2}{(\sigma + A^2)^2 + B^2 - A^4 + 2A^2B} \ge \frac{1}{2}.$$
 (48)

This restriction on *m* perfectly coincides with what was experimentally confirmed.

# 3.2. The Basic Distributions in the Random Phase Problem and Certain of Their Properties

As is mentioned above, the Rayleigh, the n- and the q-distributions are the particular solutions, and the m-distribution is a general but approximate solution, of the so-called random phase problem, and they are all identical under certain specific conditions as stated below. Their properties have been fully investigated, certain of them except those of the last being listed in the following.

(a) The Rayleigh distribution.

$$p(R) = \frac{2R}{\sigma} e^{-(R^2/\sigma)} \equiv \mathscr{L}(R, \sigma),$$

$$\overline{R^2} = \sigma, \quad \overline{R^{\nu}} = \sigma^{\nu/2} \Gamma\left(1 + \frac{\nu}{2}\right),$$

$$F_0(\lambda) = e^{-(\sigma/4)\lambda^2}, \quad \phi(z) = \frac{1}{(\sigma z + 1)}.$$

$$(49)$$

(b) The *n*-distribution.

$$p(R) = \frac{2R}{\sigma} e^{-(R^2 + R_0^2)/\sigma} I_0\left(\frac{2RR_0}{\sigma}\right) \equiv \mathcal{N}(R, R_0, \sigma),$$

$$\overline{R}^2 = \sigma + R_0^2, \quad \overline{R}^{\nu} = \sigma^{\nu/2} \Gamma\left(1 + \frac{\nu}{2}\right)_1 F_1\left(-\frac{\nu}{2}, 1; -\frac{R_0^2}{\sigma}\right),$$

$$F_0(\lambda) = e^{-(\sigma/4)\lambda^2} J_0(\lambda R_0), \quad \phi(z) = \frac{1}{(\sigma z + 1)} e^{-[z/(\sigma z + 1)]R_0^2},$$

$$\eta = \frac{R_0}{\sqrt{\sigma}}, \quad \text{Parameters:} \quad (\sigma, R_0), \quad (\Omega, \eta).$$

$$\lim_{\substack{R_0 \to 0 \\ (\eta \to 0)}} \mathcal{N}(R, R_0, \sigma) = \mathcal{L}(R, \sigma) \equiv \mathcal{M}(R, 1, \sigma). \quad (51)$$

(c) The q-distribution.

$$p(R) = \frac{2R}{\sqrt{\alpha\beta}} e^{-(R^{2}/2)(1/\alpha+1/\beta)} I_{0} \left\{ \frac{R^{2}}{2} \left( \frac{1}{\beta} - \frac{1}{\alpha} \right) \right\} \equiv \mathcal{Q}(R, \alpha, \beta),$$

$$\overline{R^{2}} = \frac{1}{2} (\alpha + \beta), \quad \overline{R^{\nu}} = \Omega^{\nu/2} \Gamma \left( 1 + \frac{\nu}{2} \right)_{2} F_{1} \left( -\frac{\nu - 2}{4}, -\frac{\nu}{4}; 1; K^{2} \right),$$

$$F_{0}(\lambda) = e^{-\lambda^{2} (\alpha + \beta)/8} I_{0} \left\{ \frac{\lambda^{2}}{8} (\alpha - \beta) \right\}, \quad \phi(z) = \frac{1}{\sqrt{(z\alpha + 1)(z\beta + 1)}},$$

$$k = \sqrt{\frac{\beta}{\alpha}}, \quad K = \frac{(\alpha - \beta)}{(\alpha + \beta)}, \quad \text{Parameters:} \quad (\alpha, \beta), (\Omega, k).$$

$$\lim_{\substack{\beta \to \alpha \\ (k \to 1)}} \mathcal{Q}(R, \alpha, \beta) = \mathcal{U}(R, \alpha) \equiv \mathcal{M}(R, 1, \alpha),$$

$$\lim_{\substack{\beta \to \alpha \\ (k \to 0)}} \mathcal{Q}(R, \alpha, \beta) = \mathcal{M}(R, \frac{1}{2}, \frac{\kappa}{2}/2).$$
(53)

# 3.3. The Interrelations between the m-Distribution and Other Basic Distributions (NAKAGAMI, WADA and FUJIMURA, 1953)

The foregoing discussions in this section maintain that the *m*-distribution must include all other basic distributions, and that if a certain part of the *m*-distribution corresponds to the *q*-distribution, the remaining part might also correspond to the *n*-distribution. These dependences are definitely described by functional relations between the parameters.

(a) The relation between the m-distribution and the n-distribution. In this case, the basic relations between the parameters are

$$\Omega = \sigma + R_0^2, \qquad \frac{1}{m} = 1 - \frac{R_0^4}{\Omega^2}, \qquad m \ge 1.$$
(54)

To represent the n-distribution in terms of the m-distribution, we have only to take the parameters as

$$\Omega = \sigma + R_0^2, \qquad m = \frac{\Omega^2}{\Omega^2 - R_0^4} = \frac{(1+\eta^2)^2}{(1+\eta^2)^2 - \eta^4}.$$
 (55)

Further, for the inverse expression, the parameter relations take the forms

$$R_0^2 = \frac{\Omega}{m} \sqrt{m^2 - m}, \qquad \sigma = \frac{\Omega}{m} (m - \sqrt{m^2 - m}), \qquad \eta = \sqrt{\frac{\sqrt{m^2 - m}}{m - \sqrt{m^2 - m}}}.$$
 (56)

(b) The relation between the m-distribution and the q-distribution. In like manner, the parameter dependences are reduced to

$$\Omega = \frac{\alpha + \beta}{2}, \qquad \frac{1}{m} = 1 + \frac{(\alpha - \beta)^2}{4\Omega^2}, \qquad \frac{1}{2} \leqslant m \leqslant 1. \tag{57}$$

Accordingly, for the transformation from the q-distribution into the m-distribution, the parameter relations are given by

$$\Omega = \frac{\alpha + \beta}{2}, \qquad m = \frac{(\alpha + \beta)^2}{(\alpha + \beta)^2 + (\alpha - \beta)^2}.$$
 (58)

Next, in the inverse transformation, the relations must be taken as

$$\alpha = \frac{\Omega}{m} (m + \sqrt{m - m^2}), \qquad \beta = \frac{\Omega}{m} (m - \sqrt{m - m^2}),$$
$$k = \sqrt{\frac{m - \sqrt{m - m^2}}{m + \sqrt{m - m^2}}}. \tag{59}$$

The errors arising from these transformations are negligible for our present purposes.

(c) General aspect of the interrelations. The above interrelations are not only of considerable importance as regards a better understanding of the situation of the *m*-distribution, but they are of much practical use in its various applications. Therefore, we show some more-detailed characteristics of these in summarized form as follows.

(1) The *m*-distribution with the parameter  $\frac{1}{2} \leq m \leq 1$  corresponds to the *q*-distribution, i.e.

$$\mathscr{M}(R, m, \Omega) \doteq \mathscr{Q}(R, \alpha, \beta), \qquad \mathscr{M}(R, \frac{1}{2}, \Omega) \equiv \mathscr{Q}(R, 2\Omega, 0).$$
(60)  
$$\overset{0 \to \beta \to \alpha}{\underset{(0 \to k \to 1)}{\overset{0 \to \beta \to \alpha}{\overset{0 \to \alpha}{\overset{0 \to \beta \to \alpha}{\overset{0 \to \alpha}{\overset{0 \to \beta \to \alpha}{\overset{0 \to \alpha}{\overset{$$

(2) The *m*-distribution with the parameter  $1 \leq m$  corresponds with the *n*-distribution, i.e.

$$\mathscr{M}(R, m, \Omega) \doteq \mathscr{N}(R, R_0, \sigma), \qquad \mathscr{M}(R, 1, \Omega) = \mathscr{N}(R, 0, \Omega).$$
(61)  
$$\overset{0 \to R_0 \to \infty}{\underset{(0 \to \eta \to \infty)}{\longrightarrow}}$$

(3) At the junction point 
$$m = 1$$
, the four distributions are all identical, i.e.  
 $\mathscr{Q}(R, \Omega, \Omega) \equiv \mathscr{M}(R, 1, \Omega) \equiv \mathscr{L}(R, \Omega) \equiv \mathscr{N}(R, 0, \Omega).$  (62)

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# 3.4. Generalized Forms of the Basic Distributions (NAKAGAMI and NISHIO, 1954b)

Now, we shall derive the generalized forms of the basic distributions stated above. These are defined as the distributions of the sums of squares of n independent m-variables.

$$R^2 = r_1^2 + r_2^2 + \ldots + r_n^2, \tag{63}$$

where  $r_i$ 's follow either one of the three basic distributions, i.e. the Rayleigh, the n- and the q-distribution.

(a) Generalized form of the Rayleigh distribution.

Case I. If  $r_i$ 's follow  $\mathscr{L}(r_i, \sigma)$ , then the distribution of R becomes the standard form of the *m*-distribution

$$p(R) = \mathscr{M}(R, m, \Omega), \tag{64}$$

where m = n,  $\Omega = n\sigma$ . Its moments,  $F_{*}(\lambda)$  and  $\phi(z)$  are expressed by

$$\overline{R}^{\overline{r}} = \frac{\Gamma\left(m + \frac{\nu}{2}\right)}{\Gamma(m)} \left(\frac{\Omega}{m}\right)^{r/2}, \quad \phi(z) = \frac{1}{\left(\frac{\Omega}{m}z + 1\right)^{m}},$$

$$F_{0}(\lambda) = e^{-(\Omega/4m)\lambda^{2}} L_{m-1}\left(\frac{\Omega}{4m}\lambda^{2}\right), \quad F_{m-1}(\lambda) = e^{-(\Omega/4m)\lambda^{2}},$$

$$= e^{-(\Omega/4)\lambda^{2}} I_{0}\left(\frac{\Omega}{4}\lambda^{2}\right) \qquad (m = \frac{1}{2}).$$
(65)

(b) Generalized form of the *n*-distribution.

Case II. If  $r_i$ 's follow  $\mathcal{N}(R, r_{0,i}, \sigma)$ , then the distribution of R becomes

$$p(R) = \frac{2R^{n}}{\sigma R_{0}^{n-1}} e^{-(R^{2} + R_{0}^{2})/\sigma} I_{n-1}\left(\frac{2RR_{0}}{\sigma}\right) \equiv \mathcal{N}_{n}(R, R_{0}, \sigma),$$
(66)

where  $R_{0^2} = \sum_{i=1}^{n} r_{0,i}^2$ , its moments,  $F_{r}(\lambda)$  and  $\phi(z)$ , being expressed by

$$\overline{R^{2}} = \sum_{i=1}^{n} (\sigma + r_{0,i}^{2}), \qquad \overline{R^{\nu}} = \sigma^{\nu/2} \frac{\Gamma\left(n + \frac{\nu}{2}\right)}{\Gamma(n)} {}_{1}F_{1}\left(-\frac{\nu}{2}, n; -\frac{R_{0}^{2}}{\sigma}\right),$$

$$\phi(z) = \frac{1}{(\sigma z + 1)^{n}} \exp\left\{-\frac{z}{(\sigma z + 1)} R_{0}^{2}\right\}, \qquad F_{n-1}(\lambda) = e^{-(\sigma/4)\lambda^{2}} \Lambda_{n-1}(\lambda R_{0}).$$

$$\left.\right\}$$
(67)

This form of distribution, named "generalized *n*-distribution", can be also derived by a procedure similar to that in section 3.1.

(c) Generalized form of the q-distribution.

Case III. If the  $r_i$ 's follow  $\mathscr{Q}(r_i, \alpha, \beta)$ , then the distribution becomes

$$p(R) = \frac{2\sqrt{\pi}R^{n}e^{-(R^{2}/2)(1/\alpha+1/\beta)}}{(\alpha\beta)^{n/2}\Gamma(\frac{n}{2})(\frac{1}{\beta}-\frac{1}{\alpha})^{(n-1)/2}}I_{(n-1)/2}\left\{\frac{R^{2}}{2}\left(\frac{1}{\beta}-\frac{1}{\alpha}\right)\right\} \equiv \mathcal{Q}_{n}(R, \alpha, \beta). \quad (68)$$

٦

Its moments,  $F_{r}(\lambda)$  and  $\phi(z)$ , being expressed by

$$\overline{R}^{\overline{2}} = n\left(\frac{\alpha+\beta}{2}\right), \, \overline{R}^{\overline{\nu}} = \frac{\Gamma\left(n+\frac{\nu}{2}\right)}{\Gamma(n)} \,\Omega_{2}^{\nu/2} F_{1}\left(-\frac{\nu-2}{4}, -\frac{\nu}{4}; \frac{n+1}{2}; K^{2}\right), \\
F_{n-1}(\lambda) = e^{-(\lambda^{2}/8)(\alpha+\beta)} \Lambda_{(n-1)/2}\left\{j\frac{\lambda^{2}}{8}(\alpha-\beta)\right\}, \, \phi(z) = \frac{1}{\{(z\alpha+1)(z\beta+1)\}^{n/2}}.$$
(69)

This form of distribution, named "generalized q-distribution", can be also derived by a procedure similar to that in section 3.1.

It is of interest to see that the above reduction procedures of the generalized forms (66) and (68) require their parameter n to be a positive integer, but the necessary and sufficient conditions of a distribution permit it to take any positive number not less than unity at least. Therefore, they should be accepted in this wider sense under the name of the generalized forms.

(d) Their interrelations in a particular case.

Case IV. Interrelations under specified conditions.

$$\lim_{\beta \to \alpha(=\sigma)} \mathscr{Q}_n(R, \alpha, \beta) = \lim_{R_0 \to 0} \mathscr{N}_n(R, R_0, \sigma) = \mathscr{M}(R, n, n\sigma).$$
(70)

(e) Further generalizations of the distributions. As is easily observed, the above distributions are further generalized based on the relation (63) as follows.

Case V. If the  $r_i$ 's follow  $\mathcal{M}(r_i, m, \Omega)$ , then the distribution of R is reducible to

$$p(R) = \mathcal{M}(R, nm, n\Omega). \tag{71}$$

Case VI. If  $r_i$ 's follow  $\mathcal{N}_{\nu_i}(r_i, R_{0,i}, \sigma)$ , then the distribution of R can also be reduced to

$$p(R) = \mathcal{N}_{r}(R, R_{0}, \sigma), \qquad (72)$$

where  $v = \Sigma v_i$ ,  $R_0^2 = \Sigma R_{0,i}^2$ .

Case VII. If  $r_i$ 's follow  $\mathcal{Q}_{r_i}(r_i, \alpha, \beta)$ , then the distribution of R can also be reduced to

$$p(R) = \mathcal{Q}_{\mu}(R, \alpha, \beta), \qquad (73)$$

where  $v = \Sigma v_i$ .

These reproducible characters of the generalized forms are of great importance in the theory of these distributions. Further, their parameter interrelations are also similar to that of the basic distributions.

#### 4. Some Further Characteristics of the *m*-Distribution

In this section, some further descriptions on the characteristics of the mdistribution, which are of great use in practical applications, are briefly given.

# 4.1. Distribution of the Sums of Squares of m-Variables (NAKAGAMI and WADA, 1953)

Here we shall consider the distributions of the sums of squares of some independent m-variables.

(a) Special case. At first, we consider the case of two *m*-variables.

Case I. Let  $r_1$  and  $r_2$  follow  $\mathcal{M}(r_1, m_1, \Omega_1)$  and  $\mathcal{M}(r_2, m_2, \Omega_2)$  respectively, then the distribution of their sum and its characteristic function become

$$p(R) = \frac{2R^{2\mu-1}e^{-(R^{2}/2)(1/\sigma_{1}+1/\sigma_{2})}}{\Gamma(2\mu)\sigma_{1}^{m_{1}}\sigma_{2}^{m_{2}}\left(\frac{1}{\sigma_{1}}-\frac{1}{\sigma_{2}}\right)^{\mu}}M_{r,\mu-\frac{1}{2}}\left\{R^{2}\left(\frac{1}{\sigma_{1}}-\frac{1}{\sigma_{2}}\right)\right\},$$

$$\phi(z) = \frac{1}{(\sigma_{1}z+1)^{m_{1}}(\sigma_{2}z+1)^{m_{2}}},$$
(74)

respectively, where  $M_{r,\mu}(x)$  is the Whittaker function, and

$$\sigma_1 = rac{\Omega_1}{m_1}\,, \qquad \sigma_2 = rac{\Omega_2}{m_2}\,, \qquad \mu = rac{m_1 + m_2}{2}\,, \qquad 
u = rac{m_1 - m_2}{2}\,.$$

This distribution plays an important role in the author's theory, and is termed by him the "M-distribution".

Case II. In a special case  $m_1 = m_2 = m$ , (74) can be reduced to a simpler form

$$p(R) = \mathcal{Q}_{2m}(R, \sigma_2, \sigma_1). \tag{75}$$

Further, in a more special case,  $m_1 = m_2 = m$  and  $\Omega_1 = \Omega_2 = \Omega$  we get the simplest form, i.e. the *m*-distribution

$$p(R) = \mathcal{M}(R, 2m, 2\Omega), \tag{76}$$

as it is readily supposed to be.

(b) General case. Next, we shall proceed to a more general case

 $R^{2} = a_{1}r_{1}^{2} + a_{2}r_{2}^{2} + \ldots + a_{n}r_{n}^{2}.$  (77)

For simplicity, the following discussions will be confined to some distributions under certain conditions in fading practice.

Case I. If  $r_i$ 's are distributed according to  $\mathcal{M}(r_i, m_i, \Omega_i)$ , and if the conditions  $\frac{\Omega_1}{m_1} = \frac{\Omega_2}{m_2} = \ldots = \frac{\Omega_n}{m_n}$  and  $a_i = 1$  are satisfied, then we have

$$p(R) = \mathscr{M}(R, \Sigma m_i, \Sigma \Omega_i).$$
(78)

Case II. If the  $r_i$ 's are distributed according to  $\mathcal{M}(r_i, m_i, \Omega_i)$ , and if the  $a_i$ 's satisfy the conditions  $a_i = \frac{\Omega_1 + \Omega_2 + \ldots + \Omega_n}{m_1 + m_2 + \ldots + m_n} \left(\frac{m_i}{\Omega_i}\right)$ , then we have  $p(R) = \mathcal{M}(R, \Sigma m_i, \Sigma \Omega_i). \tag{79}$ 

Case III. If the  $r_i$ 's are distributed according to  $\mathscr{M}\left(r_i, m_i, \frac{\Omega_i}{a_i}\right)$ , then we have approximately

where

and

$$p(R) \simeq \mathscr{M}(R, {}_{0}M, \Sigma\Omega_{i}),$$

$${}_{0}M = (\Sigma\Omega_{i})^{2}/\Sigma\left(\frac{\Omega_{i}}{\overline{m}_{i}}\right).$$
(80)

## 4.2. Distribution of the Sum of m-Variables

Next, we consider the sum of n independent variables

$$R = r_1 + r_2 + \ldots + r_n, \qquad (81)$$

where  $r_i$ 's follow the *m*-distribution  $\mathcal{M}(r_i, m_i, \Omega_i)$ . After some rather complex calculations, in this case, we arrive at

$$p(R) \simeq \mathcal{M}(R, {}_{0}m, {}_{0}\Omega)$$
(82)

approximately, where

$${}_{0}\Omega = \overline{R^{2}} = n\Omega + n(n-1)\Omega \frac{\Gamma^{2}(m+\frac{1}{2})}{m\Gamma^{2}(m)} \simeq n^{2}\Omega \left(1 - \frac{1}{5m}\right),$$

$${}_{0}m = \frac{(\overline{R^{2}})^{2}}{(R^{2} - \overline{R^{2}})^{2}} = f(m, n)mn.$$
(83)

The functional form of f(m, n) is rather complex, but the numerical values are



nearly equal to unity for all values of m and n, as illustrated in Fig. 4.1, so that the parameters in (83) are approximately reduced to the concise forms

$$_{0}m \simeq mn, \quad _{0}\Omega \simeq n^{2}\Omega.$$
 (84)

It is very important, in this case, to notice that the distribution in terms of db-intensity  $\chi$  can be closely expressed by

$$p(\chi) \simeq \frac{2(mn)^{mn}}{M\Gamma(mn)} \exp\left\{mn\left(\frac{2\chi}{M} - e^{2\chi/M}\right)\right\}.$$
(85)

This relation is of great use in practical applications, as will be shown later. It was first observed in our experiments on diversity receptions (NAKAGAMI, AKAZAWA and TANAKA, 1941), and later proved both theoretically and numerically (NAKAGAMI and SASAKI, 1942b).

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## 4.3. The Basic Characters of the Combined Output in Diversity Systems

The fading statistics in various diversity systems are uniquely determined by the cumulative distribution of the combined signal. This distribution might be supposed apparently to depend upon the kind of diversity, but essentially it is determined by the following factors: (i) Type of fading. (ii) Magnitude of diversity effect. (iii) Method of combination of component signals.

Here, we assume that the type of fading takes a form of the *m*-distribution and that the diversity effect is perfect. Under these conditions, we (e.g. NAKAGAMI, 1942a, 1942b, NAKAGAMI and WADA, 1953) obtained the statistical characters of combined output of various methods of combination, based on the foregoing formulas.

Some of them are summarized in the following, where the notations  $M(\chi, m)$  and  $M_n(\chi, m)$  stand for the cumulative distributions of a single output  $R_i$ , and the combined output R respectively,

Case I. Linear-addition method;  $R = \sum_{i=1}^{n} R_i$ .

where

$$M_{n}(\chi, m) \simeq M(\chi - g_{1}, mn),$$

$$g_{1} \simeq 20 \log_{10} n + 10 \log_{10} \left(1 - \frac{1}{5m}\right).$$
(86)

Case II. Switching method;  $R = \frac{1}{n} \Sigma R_i$ .

$$M_n(\chi, m) \simeq M(\chi, mn).$$
 (87)

Case III. Square-addition method;  $R^2 = \sum_{i=1}^{n} R_i^2$ .

$$M_n(\chi, m) = M(\chi - g_2, mn),$$

$$g_n = 10 \log m$$
(88)

where

$$g_2 = 10 \log_{10} n.$$
  
Case IV. Maximum-signal-selection method;  $R = Max [R_i].$ 

$$M_{n}(\chi, m) = \{M(\chi, m)\}^{n}.$$
(89)

As mentioned above, the cumulative distribution of combined output  $M_n(\chi, m)$  can be simply determined by  $M(\chi, m)$  of the signal. Hence, all the statistical characters in various systems of diversity reception can be readily estimated with only a sheet of the *m*-chart shown in Fig. 2.4. For the detailed process of obtaining the characters reference should be made to the original papers (Japan (NAKAGAMI), 1956). The adequacy of our method was fairly well confirmed in some observations (NAKAGAMI, AKAZAWA and TANAKA, 1941).

#### 4.4. Distributions of the Product and the Ratio of Two m-Variables

In the fading problems, the distributions of the product and the ratio of two m-variables are often required. Therefore, in the following we shall give them without proof.

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(a) Distribution of the product of two *m*-variables. Now, if  $R_1$  and  $R_2$  follow  $\mathcal{M}(R_1, m_1, \Omega_1)$  and  $\mathcal{M}(R_2, m_2, \Omega_2)$  respectively, then the distribution of their product  $R_p = R_1 R_2$  is given by

$$p(R_{p}) = \frac{4R_{p}^{m_{1}+m_{2}-1}}{\Gamma(m_{1})\Gamma(m_{2})} \left(\sqrt{\frac{m_{1}m_{2}}{\Omega_{1}\Omega_{2}}}\right)^{m_{1}+m_{2}} K_{m_{1}-m_{2}} \left(2\sqrt{\frac{m_{1}m_{2}}{\Omega_{1}\Omega_{2}}} R_{p}\right), \tag{90}$$

where  $K_{\nu}(x)$  is the modified Bessel function of the second kind.

Using the transformation  $X_p = \sqrt{\frac{m_1 m_2}{\Omega}} R_p$ , where  $\Omega = \overline{R_p^2}$ , we get a simpler form

$$p(X_{p}) = \frac{4X_{p}^{m_{1}+m_{2}-1}}{\Gamma(m_{1})\Gamma(m_{2})} K_{m_{1}-m_{2}}(2X_{p}).$$
(91)

These properties of this distribution were discussed in some detail (NAKAGAMI and  $\overline{O}_{TA}$ , 1957).

(b) Distribution of the ratio of two *m*-variables. Next, let  $R_1$  and  $R_2$  follow  $\mathcal{M}(R_1, m_1, \Omega_1)$  and  $\mathcal{M}(R_2, m_2, \Omega_2)$  respectively, then the distribution of the ratio  $F = \frac{R_1^2}{\Omega_1} / \frac{R_2^2}{\Omega_2}$  is reducible to  $\Gamma(m_1 + m_2) / (m_1)^{m_1} = (m_1 + m_2)^{-(m_1 + m_2)}$ 

$$p(F) = \frac{\Gamma(m_1 + m_2)}{\Gamma(m_1)\Gamma(m_2)} \left(\frac{m_1}{m_2}\right)^{m_1} F^{m_1 - 1} \left(1 + \frac{m_1}{m_2}F\right)^{-(m_1 + m_2)}.$$
(92)

This form of the distribution is the same as the well-known "F-distribution" due to SNEDECOR, but the marked difference in their parameters should be remembered as previously mentioned. The above formula is due to  $\overline{O}TA$  (1956).

#### 4.5. Intensity Distribution Due to Random Interferences

Now, we shall take up a more general type of interference, in which the amplitudes and phases of component waves are mutually independently distributed according to certain statistical laws. The resultant intensity in this case, of course, may be expressed by (41).

(a) Case of a large number of component waves (NAKAGAMI and TANAKA, 1951). For a large number of component waves we may assume the central limit theorem, i.e. the Gaussian distribution of the components x and y given in (41). Under this assumption we are able to reduce the intensity distribution p(R), after some calculations, to

$$p(R) = \frac{2R}{\sqrt{\alpha\beta}} e^{-(gR^2 + hA^2)} \sum_{n=0}^{\infty} \epsilon_n I_n \left\{ \frac{R^2}{2} \left( \frac{1}{\beta} - \frac{1}{\alpha} \right) \right\} I_{2n}(2R\sqrt{p^2 + q^2}) \cos n\Theta, \quad (93)$$

where  $\epsilon_n$  is Neumann's factor, and

q

$$g = rac{1}{2} \left( rac{1}{lpha} + rac{1}{eta} 
ight), \qquad h = p^2 lpha + q^2 eta, \qquad p = rac{A}{lpha} \cos{\left(\delta_1 - \delta_2
ight)},$$
  
 $= rac{A}{eta} \sin{\left(\delta_1 - \delta_2
ight)}, \qquad an \Theta = rac{lpha}{eta} an{\left(\delta_1 - \delta_2
ight)}, \qquad lpha = \sigma + B, \qquad eta = \sigma - B,$ 

A, B,  $\sigma$ ,  $\delta_1$ , and  $\delta_2$  being the same as in (47).

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In the particular cases  $\alpha = \beta$  and A = 0, (93) yields the exact forms of the *n*- and the *q*-distributions, respectively. The characteristics of this distribution were fully discussed above. Of course, it also approximates, within a small error, to  $\mathcal{M}(R, m, \sigma + A^2)$  if *m* is given by (47).

(b) Case of arbitrary number of component waves (NAKAGAMI, NISHIO and YOKOI, 1954). Assuming first that the phases  $\theta_i$ 's are uniformly distributed over the range  $(0, 2\pi)$ , we are able to express p(R), without any restriction, as

$$p(R) = R \int_0^\infty \lambda J_0(\lambda R) \prod_{i=1}^n J_0(\lambda r_i) \, \mathrm{d}\lambda.$$
(94)

Next, if we make the assumption that the amplitudes  $r_i$ 's follow  $\mathcal{M}(r_i, m_i, \Omega_i)$  independently, then  $F_0(\lambda)$  may be reduced to the form

$$F_{0}(\lambda) = \prod_{i=1}^{n} \left\{ e^{-(\Omega_{i}/4m)\lambda^{2}} L_{m_{i}-1}\left(\frac{\Omega_{i}}{4m_{i}}\lambda^{2}\right) \right\}.$$
(95)

In the simplest case  $m_i = 1$ , i.e. the Rayleigh distribution,  $F_0(\lambda)$  takes a simple exponential form, and p(R) also becomes  $\mathscr{L}(R, \Sigma\Omega_i)$ , as is well known. In the other cases, it seems to be difficult in general to express p(R) in a compact form with known functions. However, the general discussions in Section 3.1 enable us to approximate to it as follows:

 $p(R) \sim \mathcal{M}(R, m, \Sigma \Omega)$ 

where

$${}_{0}m = \frac{(\Sigma\Omega_{i})^{2}}{\sum_{i=1}^{n} \left(\frac{\Omega_{i}^{2}}{m_{i}}\right) + 2\sum_{i\neq j}^{n} \Omega_{i}\Omega_{j}}.$$

The approximation in the above expression improves as n and m increase, and is good enough in fading practice for all values of n and m.

For the special case,  $\Omega_i = \Omega$ ,  $m_i = m$  the parameter m reduces to

$$_{0}m = \frac{1}{1 - \frac{1}{n} + \frac{1}{nm}}$$
 (97)

From this, we can arrive at very important conclusions: (i) As  $n \to \infty$ , always  $_0m \to 1$ . (ii) When m > 1, always  $m > _0m > 1$ . (iii) When  $1 > m \ge \frac{1}{2}$ , always  $1 > _0m > m$ . (iv) When m = 1,  $_0m = 1$  always. These conclusions are of great use not only in better understanding the fading mechanism, but also in practical applications.

## 4.6. Effects of the Parameter Variations on the m-Distribution (Japan (NAKAGAMI), 1955)

We shall now confine our attention to the effects on the *m*-distribution caused by the fluctuations of its parameters m and  $\Omega_0$ . These effects are of considerable importance in the estimation of the distribution over a long term, where the parameters can no longer be considered as constant.

(96)

Now, let  $p(\tau_0, m)$  be the joint distribution of the parameters  $\tau_0$  and m, then the distribution of  $\tau$ , according to (14), can be written as

$$p(\tau) = \int_{\frac{1}{2}}^{\infty} \mathrm{d}m \int_{-\infty}^{\infty} \mathscr{M}_{\tau}(\tau, m, \tau_0) p(\tau_0, m) \,\mathrm{d}\tau_0.$$
(98)

Our recent observations and some calculations seem to support strongly that

$$p(\tau_0, m) = p(\tau_0) p(m),$$

$$p\left(\frac{1}{m}\right) = \frac{1}{\sqrt{2\pi}A} e^{-(1/2A^2)\{(1/m - 1/m_0)\}^2}, \qquad p(\tau_0) = \frac{1}{\sqrt{2\pi}\sigma_0} e^{-(1/2\sigma_0^2)\{(\tau_0 - \bar{\tau}_0)\}^2}.$$
(99)

But these relations, with the exception of the last, are not yet established.

Therefore, in the present discussion, only the effect of  $\tau_0$  or  $\Omega$  will be considered. After some calculation, we get a final expression

$$p(\tau) \simeq C \mathscr{M}_{\tau}(\tau, m, \bar{\tau}_0) S(\tau, m, \bar{\tau}_0), \qquad (100)$$

٦

where

$$S(\tau, m, \bar{\tau}_0) = \frac{M}{\sqrt{M^2 + 4\sigma_0^2 m Q}} \exp\left\{\frac{2\sigma_0^2 m^2 (1-Q)^2}{M^2 + 4\sigma_0^2 m Q}\right\},$$

 $Q = e^{(2/M)(\tau - \tau_0)}, C =$  Normalizing factor, nearly equal to 1.

Numerical calculations (NAKAGAMI, TANAKA and KANEHISA, 1957, see Figs. 4.3 and 4.4) clearly indicate the remarkable tendency that, with the increase in fluctuations of  $\tau_0$  or  $\Omega$ ,  $p(\tau)$  gradually approaches a log-normal type of distribution. For example, even in the extreme case of m equal to  $\frac{1}{2}$ ,  $p(\tau)$  may be taken as a log-normal form for larger values of  $\sigma_0$  than 10 db, and the same will hold for the Rayleigh distribution for values of  $\sigma_0$  beyond 7 db.

These properties of the *m*-distribution apparently account for GROSSKOPF's (1953) observations.

#### 4.7. Some More General Forms of the m-Distribution

In general, we may obtain in various ways many other forms of distribution which are of a more general nature than the generalized forms of the m-distribution previously described. We next exhibit two such forms.

One form is

$$p(R) = \frac{2\rho R}{I_{\alpha-1}\left(\frac{\lambda-\mu}{\rho}\right)} e^{-\left\{(1/\rho)(\lambda+\mu)+\rho R^2\right\}} I_{\alpha-1}\left\{\sqrt{2}R(\sqrt{\lambda}+\sqrt{\mu})\right\} \times I_{\alpha-1}\left\{\sqrt{2}R(\sqrt{\lambda}-\sqrt{\mu})\right\}, \quad (101)$$

where  $\alpha$ ,  $\lambda$ ,  $\rho$  and  $\mu$  are the parameters, being expressed in terms of the moments. In the limit  $\lambda \rightarrow \mu$ , it reduces to the generalized *n*-distribution. Its characteristic function is given by

$$\phi(z) = \frac{\rho e^{-z(\lambda+\mu)/\rho(z+\rho)}}{I_{\alpha-1}\left(\frac{\lambda-\mu}{\rho}\right)(z+\rho)} I_{\alpha-1}\left(\frac{\lambda-\mu}{z+\rho}\right).$$
(102)

Another form is

$$p(R) = \frac{2\Gamma(2\nu)(R^2\rho)^{\alpha-\nu}}{\Gamma(\alpha)\sqrt{\lambda}M_{\nu-\alpha,\nu-\frac{1}{2}}\left(\frac{1}{\rho\lambda}\right)} e^{-\left\{(1/2\rho\lambda+\rho R^2)\right\}} I_{2\nu-1}\left(\frac{2R}{\sqrt{\lambda}}\right),$$
(103)

where  $\alpha$ ,  $\nu$ ,  $\rho$  and  $\lambda$  are the parameters. In the special case  $2\nu = \alpha$ , this also reduces to the generalized *n*-distribution. And its characteristic function becomes

$$\phi(z) = \frac{\rho^{\alpha-\nu} e^{-z/2\lambda\rho(z+\rho)}}{(z+\rho)^{\alpha-\nu} M_{\alpha-\nu,\nu-\frac{1}{2}} \left(\frac{1}{\rho\lambda}\right)} M_{\alpha-\nu,\nu-\frac{1}{2}} \left(\frac{1}{\lambda(z+\rho)}\right).$$
(104)

Generalizations such as these might describe much wider varieties of distributions in actual fading. They are of much theoretical interest, but of less practical importance because of their formal complexities. And we shall give no further discussions on these subjects.

## 5. METHODS OF APPROXIMATING A GIVEN DISTRIBUTION FUNCTION WITH THE *m*-DISTRIBUTION

There often arise, in theoretical treatments of fading, strong demands for suitable methods of approximating to a given distribution with a specified distribution function, especially with the m-formula. These methods may be found in some different ways. Some of them are given in the sequel.

## 5.1. Methods Based on the Laguerre Polynomial Expansion (NAKAGAMI, TANAKA and KANEHISA, 1957)

Generally speaking, the distribution function of a variate defined in the positive range is usually expansible in terms of the Laguerre polynomials. Making use of this form of expansion, we are able to establish the method of approximating to a given distribution of any positive variate by means of the m-distribution. On this form of expansion we give some brief accounts in the following.

(a) A more general form of the expansion. In the first place, we shall derive in a more general manner a form of expansion in terms of the Laguerre polynomials, without rigorous discussions.

In virtue of the known formula (e.g. ERDÉLYI, 1953a)

$$e^{t}(xt)^{-\frac{d}{2}}J_{\alpha}(2\sqrt{xt}) = \sum_{n=0}^{\infty} \frac{L_{n}^{\alpha}(t)}{\Gamma(n+\alpha+1)} t^{n}, \qquad (105)$$

 $F_{\nu}(\lambda)$  in (32) may be expanded in the form

$$F_{\nu}(\lambda) = \Gamma(m) e^{-(\Omega/4m)\lambda^2} \sum_{n=0}^{\infty} \frac{L_n^{m-1}\left(\frac{m}{\Omega}\xi^2\right)}{\Gamma(n+m)} \left\{\frac{\lambda^2}{4}\frac{\Omega}{m}\right\}^n, \qquad (106)$$

where  $\Omega = \overline{\xi^2}$ ,  $m = \nu + 1 = \frac{\Omega^2}{(\xi^2 - \overline{\xi^2})^2}$ , as usual.

Evaluating (31), after substitution into it from (105), we can get the required expansion

$$p(R) = \mathscr{M}(R, m, \Omega) \sum_{n=0}^{\infty} \frac{n! \Gamma(m)}{\Gamma(n+m)} L_n^{m-1} \left(\frac{m}{\Omega} \xi^2\right) L_n^{m-1} \left(\frac{m}{\Omega} R^2\right).$$
(107)

This formula not only affords a general formulation of the expansion in terms of the Laguerre polynomials, but clearly indicates the underlying principle of this method of approximation. Resorting to this formula we are able to obtain, at

least in form, the required approximation by only taking the average  $\overline{L_n^{m-1}\left(\frac{m}{\Omega}\,\xi^2\right)}$ .

(b) A method of expansion by means of  $F_{\nu}(\lambda)$ . For the following two methods, we shall only give the reduction process, for the underlying principles and some examples of these methods have already appeared in the foregoing discussions.

At the first step, using a given distribution, calculate  $F_{\nu}(\lambda)$  according to (32), and expand it in the form

$$F_{r}(\lambda) = e^{-(\Omega/4m)\lambda^{2}} \left\{ 1 + \sum_{n=3}^{\infty} a_{n} \lambda^{2n} \right\}.$$
(108)

Next, in virtue of the formula (e.g. ERDÉLYI, 1953b)

$$e^{-x}x^{\alpha/2}L_n^{\alpha}(x) = \frac{1}{n!}\int_0^\infty e^{-t}t^{n+(\alpha/2)}J_{\alpha}(2\sqrt{xt}) dt, \qquad (\alpha > -1),$$

calculate p(R) according to (31), then we can arrive at the final expression

$$p(R) = \mathscr{M}(R, m, \Omega) \left\{ 1 + \sum_{n=3}^{\infty} 2^{2n} a_n n! \left(\frac{m}{\Omega}\right)^n L_n^{m-1}\left(\frac{m}{\Omega}R^2\right) \right\}.$$
 (109)

(c) A method of expansion by means of  $\phi(z)$ . First, with the aid of BURMANN's expansion theorem, expand  $\phi(z)$  in the form

$$\phi(z) = \frac{1}{\left(\frac{\Omega}{m}z+1\right)^m} \left(1 + \sum_{n=3}^{\infty} \frac{b_n}{n!} \left(\frac{z}{\Omega}z+1\right)^n\right).$$
(110)

Next, using the known formula (e.g. ERDÉLYI, 1954)

$$\frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} e^{zt} \frac{z^n}{(z+1)^{n+1+\nu}} \, \mathrm{d}z = \frac{\Gamma(n+1)}{\Gamma(n+\nu+1)} \, t^{\nu} e^{-t} L_n^{\nu}(t), \tag{111}$$

calculate p(R) according to (37). Then we finally have the required expansion

$$p(R) = \mathscr{M}(R, m, \Omega) \left\{ 1 + \sum_{n=3}^{\infty} b_n \frac{\Gamma(m)}{\Gamma(m+n)} L_n^{m-1}\left(\frac{m}{\Omega} R^2\right) \right\},$$
(112)

which is of course equivalent to (109).

These forms of expansion are of much theoretical interest, but they are rather inconvenient for numerical calculations, the reasons for which will be found in the following descriptions.

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The *m*-distribution—a general formula of intensity distribution of rapid fading

#### 5.2. A Method of Expansion in Terms of the m-Distribution

Before proceeding to the particular types of expansion in question, we shall show an illustrative example suggesting the underlying principle as well as the remarkable properties of this method.

(a) An illustrative example. Now, for the sake of simplicity, we take up the n-distribution. Its characteristic function is, as shown in (50), expressed by

$$\phi(z) = \frac{1}{(z\sigma+1)} e^{-\{z/(\sigma z+1)\}R_0^2}.$$
 (113)

Here if we expand the exponent as

$$\phi(z) = \frac{1}{(\sigma z + 1)} \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} \left(\frac{zR_0^2}{\sigma z + 1}\right)^n,$$

then it yields immediately from (111) the Laguerre polynomial expansion

$$p(R) = \frac{2R}{\sigma} e^{-(R^2/\sigma)} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} L_n\left(\frac{R^2}{\sigma}\right) \left(\frac{R_0^2}{\sigma}\right)^n.$$
(114)

In this series, the terms alternate in sign and the polynomials also oscillate in value. Due to these undesirable properties, this form of series is inconvenient for numerical calculations.

On the other hand, if we expand  $\phi(z)$ , after a slight modification of the exponent, in the form

$$\phi(z) = e^{-(R_0^2/\sigma)} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{z\sigma+1}\right)^{n+1} \left(\frac{R_0^2}{\sigma}\right)^n, \quad (115)$$

then by virtue of its inverse transform, we may get a particular form of series

$$p(R) = e^{-(R_0^2/\sigma)} \sum_{n=0}^{\infty} \frac{1}{n!} \mathscr{M}(R, n + 1, \sigma(n + n)) \left(\frac{R_0^2}{\sigma}\right)^n.$$
(116)

This form of series, as is evident, has the following distinguished properties. (i) It consists of a family of the *m*-distribution. (ii) The series consists only of positive terms. These properties are best utilized to advantage in the approximation of a given distribution function; e.g. they enable us to obtain the required approximation within a small error, using only a sheet of *m*-chart shown in Fig. 2.4.

(b) Expansion of certain more general distributions. In a similar manner, the generalized n- and the generalized q-distributions are readily expansible as

$$\mathcal{N}_{\nu}(R, R_0, \sigma) = e^{-(R_0^2/\sigma)} \sum_{n=0}^{\infty} \frac{\eta^{2n}}{n!} \mathcal{M}(R, \nu + n, \sigma(\nu + n)), \qquad (117)$$

where  $\eta^2 = rac{{R_0}^2}{\sigma}$  , as usual, and

$$\mathscr{Q}_{2\nu}(R,\alpha,\beta) = \frac{2^{2\nu}(\alpha\beta)^{\nu}}{(\alpha+\beta)^{2\nu}} \sum_{n=0}^{\infty} \frac{(\nu)_n}{n!} K^{2n} \mathscr{M}(R, 2(\nu+n), 2\Omega_{\nu}(\nu+n)), \quad (118)$$

where  $(\nu)_n = \nu(\nu + 1) \dots (\nu + n - 1)$ ,  $K = \frac{\alpha - \beta}{(\alpha + \beta)}$  and  $\Omega_p = \frac{2\alpha\beta}{\alpha + \beta}$ , respectively.

More general distributions (101) and (103) are also expanded in this form, as is easily seen from their respective characteristic functions.

In the case of the distribution (93), its characteristic function may be written as

$$\phi(z) = \frac{1}{\sqrt{(z\alpha + 1)(z\beta + 1)}} e^{(\alpha p^2)/(z\alpha + 1) + (\beta q^2)/(z\beta + 1) - (p^2\alpha + q^2\beta)},$$
 (119)

which indicates again that this distribution may also be expanded in the form of a positive term series similar to (118). But, in this case, a less complex positive term series in terms of the M-distribution (74) can be readily obtained from (119).

In conclusion, from the foregoing discussions and some general considerations on fading mechanisms, the above forms of positive term series, which are constructed with the families of certain specified distributions, might well describe the intensity distributions under most conditions in actual fading.

## 6. JOINT DISTRIBUTION OF TWO *m*-VARIABLES AND CERTAIN OF ITS PROPERTIES

We are now in a position to derive the joint distribution of two variables each of which follows the *m*-distribution, and further to discuss the basic characteristics of the distribution.

## 6.1. Special but Exact Distribution (NAKAGAMI, TANAKA and KANEHISA, 1957)

In order to derive the exact distribution, we start with the following relations

$$R_1^2 = r_{1,1}^2 + r_{1,2}^2 + \ldots + r_{1,n}^2, \qquad R_2^2 = r_{2,1}^2 + r_{2,2}^2 + \ldots + r_{2,n}^2.$$
(120)

(a) Derivation from the Rayleigh distribution. Now, let  $r_{1,i}$ 's and  $r_{2,i}$ 's follow  $\mathscr{L}(r_{1,i}, \Omega_1)$  and  $\mathscr{L}(r_{2,i}, \Omega_2)$  respectively, then, as is already proved,  $R_1$  and  $R_2$ follow  $\mathcal{M}(R_1, n, n\Omega_1)$  and  $\mathcal{M}(R_2, n, n\Omega_2)$  respectively. And further let

$$\frac{c(r_{1,i}^2, r_{2,j}^2)}{\sqrt{\nabla(r_{1,i}^2)\nabla(r_{2,j}^2)}} = \rho_2 \qquad (i = j) \\= 0 \qquad (i \neq j),$$
(121)

then, as is already discussed (NAKAGAMI and SASAKI, 1943), the joint distribution of  $r_{1,i}$  and  $r_{2,i}$  takes the form

$$p(r_{1,i}, r_{2,j}) = \frac{4r_{1,i}r_{2,j}}{\Omega_1\Omega_2(1-\rho_2)} e^{-(\Omega_2 r_{1,i}^2 + \Omega_1 r_2^3, j)/(\Omega_1\Omega_2(1-\rho_2))} I_0 \left\{ \frac{2r_{1,i}r_{2,j}\sqrt{\rho_2}}{\sqrt{\Omega_1\Omega_2}(1-\rho_2)} \right\} \quad (i=j)$$

$$= \mathscr{L}(r_{1,i}, \Omega_1) \mathscr{L}(r_{2,j}, \Omega_2) \qquad (i \neq j), \tag{122}$$

where  $\Omega_1 = \overline{r_{1,i}^2}$ ,  $\Omega_2 = \overline{r_{2,i}^2}$  (i = 1, 2, ..., n). From this, the characteristic function  $\phi(z_1, z_2)$  can be reduced to,

$$\phi(z_1, z_2) = \frac{1}{c^n [(z_1 + \alpha_1)(z_2 + \alpha_2) - \gamma^2]^n}, \qquad (123)$$

where

$$\frac{1}{c} = \frac{1}{\Omega_1 \Omega_2 (1 - \rho_2)}, \qquad \alpha_1 = \frac{\Omega_2}{\Omega_1 \Omega_2 (1 - \rho_2)}, 
\gamma = \frac{\sqrt{\rho_2}}{\sqrt{\Omega_1 \Omega_2 (1 - \rho_2)}}, \qquad \alpha_2 = \frac{\Omega_1}{\Omega_1 \Omega_2 (1 - \rho_2)},$$
(124)

By virtue of the formula (VOELKER and DOETSCH, 1950a)

$$\left(\frac{1}{2\pi j}\right)^{2} \int_{c-j\infty}^{c+j\infty} \int_{c-j\infty}^{c+j\infty} e^{(z_{1}x+z_{2}y)} \frac{\mathrm{d}z_{1} \,\mathrm{d}z_{2}}{[(z_{1}+\alpha_{1})(z_{2}+\alpha_{2})-\gamma^{2}]^{\nu}} \\ = \frac{(xy)^{(\nu-1)/2}}{\Gamma(\nu)\gamma^{\nu-1}} e^{-(\alpha_{1}x+\alpha_{2}y)} I_{\nu-1}(2\gamma\sqrt{xy}), \quad (125)$$

we can arrive at the required distribution

$$p(R_1, R_2) = \frac{4(R_1R_2)^n e^{-(\Omega_2R_1^2 + \Omega_1R_2^2)/(\Omega_1\Omega_2(1-\rho_2))}}{\Gamma(n)\Omega_1\Omega_2(1-\rho_2)(\sqrt{\Omega_1\Omega_2\rho_2})^{n-1}} I_{n-1} \left(\frac{2\sqrt{\rho_2}R_1R_2}{\sqrt{\Omega_1\Omega_2}(1-\rho_2)}\right)$$
  
$$= \mathcal{M}(R_1, n, n\Omega_1; R_2, n, n\Omega_2|\rho_2), \qquad (126)$$

where  $n\Omega_1 = {}_0\Omega_1 = \overline{R_1^2}$ ,  $n\Omega_2 = {}_0\Omega_2 = \overline{R_2^2}$ . As to  $\rho_2$ , it is of interest to see that

$$\frac{c(R_1^2, R_2^2)}{\sqrt{\nabla(R_1^2)\nabla(R_2^2)}} = \rho_2 = \frac{c(r_{1,i}^2, r_{2,i}^2)}{\sqrt{\nabla(r_{1,i}^2)\nabla(r_{2,i}^2)}} \qquad (i = 1, 2, \dots, n).$$
(127)

In the above discussions, n is restricted to a positive integer, but even if n is assumed to be any positive number not less than  $\frac{1}{2}$ , the formula (126) satisfies the necessary and sufficient conditions to be a joint distribution function. Therefore, it can be extended to a more general case, where n stands for a positive number not less than  $\frac{1}{2}$ .

(b) Derivation from the *m*-distribution. Taking (126) as a basis, we shall also derive a more general form of the distribution.

Now, assuming that

$$p(r_{1,i}, r_{2,j}) = \mathcal{M}(r_{1,i}, \nu, \Omega_1; r_{2,j}, \nu, \Omega_2 | \rho_2) \qquad (i = j) \\ = \mathcal{M}(r_{1,i}, \nu, \Omega_1) \mathcal{M}(r_{2,j}, \nu, \Omega_2) \qquad (i \neq j),$$
 (128)

we obtain, in like manner, the characteristic function

$$\phi(z_1, z_2) = \left(\frac{1}{c^{\nu}\{(z_1 + \alpha_1)(z_2 + \alpha_2) - \gamma^2\}^{\nu}}\right)^n.$$
(129)

And by the aid of (125), we can readily get

$$p(R_1, R_2) = \mathscr{M}(R_1, n\nu, n\Omega_1; R_2, n\nu, n\Omega_2 | \rho_2).$$
(130)

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# 6.2. General but Approximate Distribution (e.g. NAKAGAMI and NISHIO, 1953, 1955)

Further, we take up a more general case

$$R_1 = \xi_1(x_1, x_2, \ldots, x_n), \qquad R_2 = \xi_2(x_1, x_2, \ldots, x_n), \qquad (131)$$

where  $\xi_1$  and  $\xi_2$  are given functions of random variables  $x_1, x_2, \ldots, x_n$ . Even in this case, the characteristic function can be reduced, in a similar way, to

$$\phi(z_1, z_2) \simeq \frac{1}{\left\{ \left( \frac{\mathbf{0}\Omega_1 z_1}{m} + 1 \right) \left( \frac{\mathbf{0}\Omega_2 z_2}{m} + 1 \right) - \rho_2 \frac{\mathbf{0}\Omega_1 \mathbf{0}\Omega_2}{m^2} z_1 z_2 \right\}},$$
(132)

where

$$m = \frac{\overline{(\xi^2)^2}}{(\xi^2 - \overline{\xi^2})^2}, \qquad \rho_2 = \frac{c(\xi_1^2, \xi_2^2)}{\sqrt{\overline{V}(\xi_1^2)\overline{V}(\xi_2^2)}} = \frac{mc(\xi_1^2, \xi_2^2)}{{}_0\Omega_{1\,0}\Omega_2}, \qquad {}_0\Omega_1 = \overline{\xi_1^2}, \\ {}_0\Omega_2 = \overline{\xi_2^2}. \end{cases}$$
(133)

Here, if we use the formula (VOELKER and DOETSCH, 1950b)

$$\begin{split} \frac{1}{\Gamma(s)(b+1)} \int_0^\infty & \int_0^\infty e^{-(\mu x + \nu y)} \times e^{-(x+y)/(b+1)} \left(\frac{xy}{b}\right)^{(s-1)/2} J_{s-1}\left(\frac{2\sqrt{xy}}{b+1}\right) \, \mathrm{d}x \, \mathrm{d}y \\ &= \frac{1}{\{(\mu+1)(\nu+1) + b\mu\nu\}^s} (|b| < 1), \\ (|b| < 1), \\ (R(s) > 0), \end{split}$$

we can readily arrive at the final result

$$p(R_1, R_2) \simeq \mathscr{M}(R_1, m_1, {}_{0}\Omega_1; R_2, m_2, {}_{0}\Omega_2|\rho_2).$$
(134)

6.3. Some Properties of 
$$\mathcal{M}(R_1, m_1, \Omega_1; R_2, m_2, \Omega_2|\rho_2)$$
  
(NAKAGAMI and NISHIO, 1953, 1955)

The properties of this distribution have been fully discussed. Certain of them are summarized below:

(a) Expansions of the distribution function. In virtue of the Hill-Hardy formula (ERDÉLYI, 1953c)

$$(1-t)^{-1}\exp\left(-t\frac{x+y}{1-t}\right)(xyt)^{-\alpha/2}I_{\alpha}\left\{2\frac{\sqrt{xyt}}{1-t}\right\}$$
$$=\sum_{n=0}^{\infty}\frac{n!}{\Gamma(n+\alpha+1)}L_{n}^{\alpha}(x)L_{n}^{\alpha}(y)t^{n}, \quad (|t|<1, \alpha>-1),$$

we can readily expand the distribution function in a form of series

$$\mathcal{M}(R_{1}, m, \Omega_{1}; R_{2}, m, \Omega_{2}|\rho_{2}) = \mathcal{M}(R_{1}, m, \Omega_{1})\mathcal{M}(R_{2}, m, \Omega_{2})$$

$$\times \sum_{n=0}^{\infty} \frac{n! \Gamma(m)}{\Gamma(m+n)} \rho_{2}^{n} L_{n}^{m-1} \left(\frac{mR_{1}^{2}}{\Omega_{1}}\right) L_{n}^{m-1} \left(\frac{mR_{2}^{2}}{\Omega_{2}}\right), \quad (|\rho_{2}| < 1), \quad (135)$$

where  $L_n^{\nu}(x)$  is the generalized Laguerre polynomial, as usual.

Next, we shall show the expansion in terms of the m-distribution. Expanding the characteristic function (123) in the form

$$\begin{split} \phi(z_1, z_2) &= \frac{1}{c^m \{(z_1 + \alpha_1)(z_2 + \alpha_2)\}^m} \Big\{ 1 + \frac{m\gamma^2}{(z_1 + \alpha_1)(z_2 + \alpha_2)} \\ &+ \frac{m(m+1)}{2!} \frac{(\gamma^2)^2}{\{(z_1 + \alpha_1)(z_2 + \alpha_2)\}^2} + \dots \Big\}, \end{split}$$

and referring to (65), we finally get the required expansion

$$p(R_1, R_2) = (1 - \rho_2)^m \sum_{n=0}^{\infty} \frac{(m)_n}{n!} \rho_2^n \\ \times \mathcal{M}(R_1, m+n, (m+n)\Omega_1') \mathcal{M}(R_2, m+n, (m+n)\Omega_2'). \quad (136)$$

where  $(m)_n = m(m + 1) \dots (m + n - 1)$ ,  $\Omega_1' = (1 - \rho_2)\Omega_1$ ,  $\Omega_2' = (1 - \rho_2)\Omega_2$ . (b) Covariances and correlation coefficients.

$$\overline{R_1^n R_2^l} = \left(\frac{\Omega_1}{m}\right)^{n/2} \left(\frac{\Omega_2}{m}\right)^{l/2} \frac{\Gamma\left(m + \frac{n}{2}\right) \Gamma\left(m + \frac{l}{2}\right)}{\Gamma^2(m)} {}_2F_1\left(-\frac{n}{2}, -\frac{l}{2}; m; \rho_2\right), \quad (137)$$

$$c(R_1^n, R_2^n) = \left(\frac{\Omega_1\Omega_2}{m^2}\right)^{n/2} \frac{\Gamma^2\left(m + \frac{n}{2}\right)}{\Gamma^2(m)} \left\{ {}_2F_1\left(-\frac{n}{2}, -\frac{n}{2}; m; \rho_2\right) - 1 \right\}, \quad (138)$$

$$\rho_{n} = \frac{C(R_{1}^{n}, R_{2}^{n})}{\sqrt{\nabla(R_{1}^{n})\nabla(R_{2}^{n})}} = \frac{\Gamma^{2}\left(m + \frac{n}{2}\right)}{\Gamma(m)\Gamma(m + n) - \Gamma^{2}\left(m + \frac{n}{2}\right)} \left\{ {}_{2}F_{1}\left(-\frac{n}{2}, -\frac{n}{2}; m; \rho_{2}\right) - 1 \right\} \simeq \rho_{2}. \quad (139)$$

(c) Amplitude characteristic function. In two different manners,  $F_{\nu}(\lambda)$  is reducible to

$$F_{m-1}(\lambda) = e^{-(1/4m)(\Omega_1\lambda_1^2 + \Omega\lambda_2^2)} \Lambda_{m-1} \left\{ j \left( \frac{\lambda_1\lambda_2 \sqrt{\rho_2 \Omega_1 \Omega_2}}{2m} \right) \right\}.$$
(140)

Based upon the above relations, we (NAKAGAMI and NISHIO, 1955) established the unified theory of diversity effects, and also discussed in some detail the dependence of these effects on the coherency of the waves.

## 6.4. The Distribution of the Sum of Squares of Two Correlated m-Variables (NAKAGAMI and NISHIO, 1955)

Now, we consider the distribution of the sum of squares of two variables which follow (130). The characteristic function in this case can be reduced to

$$\phi(z) = \frac{1}{[\sigma_1 \sigma_2 (1 - \rho_2) \{ (z + \alpha)^2 - \beta^2 \}]^m},$$
(141)

where

$$\alpha = \frac{\sigma_1 + \sigma_2}{2\sigma_1\sigma_2(1 - \rho_2)}, \qquad \beta^2 = \frac{(\sigma_1 - \sigma_2)^2 + 4\sigma_1\sigma_2\rho_2}{4\sigma_1^2\sigma_2^2(1 - \rho_2)^2}, \qquad \sigma_1 = \frac{\Omega_1}{m}, \qquad \sigma_2 = \frac{\Omega_2}{m}.$$

From this, we obtain the final result

$$p(R) = \frac{2R\sqrt{\pi}}{\{\sigma_1\sigma_2(1-\rho_2)\}^m\Gamma(m)} e^{-(\sigma_1+\sigma_2)R^2/(2\sigma_1\sigma_2)(1-\rho_2)} \left(\frac{R^2}{2\beta}\right)^{m-\frac{1}{2}} I_{m-\frac{1}{2}}(\beta R^2)$$
(142)

i.e. a type of the generalized q-distribution.

For a small value of R, p(R) can be approximately written as

$$p(R) \simeq \frac{1}{S^m} \{p(R)\}_{\rho_2=0},$$
 (143)

where  $S = (1 - \rho_2)$ .

These formulas will be found to be useful in some practical applications, e.g. in a dual diversity reception by the square-addition method they afford the means of estimating the degree of improvement available from this method of diversity, the diversity effect being reasonably expressed by the quantity S in (143).

## 6.5. Distributions of the Product and the Ratio of Two Correlated m-Variables

Before concluding, we shall add two distribution formulas of the product and the ratio of two correlated variables  $R_1$  and  $R_2$  which follow  $\mathcal{M}(R_1, m, \Omega_1;$  $R_2, m, \Omega_2|\rho_2).$ 

The distribution of the product  $R_v = R_1 R_2$  is given by NAKAGAMI and  $\overline{O}$ TA (1957) as follows:

$$p(x_{p}) = \frac{4(1-\rho_{2})^{m}}{\Gamma(m)(\sqrt{\rho_{2}})^{m-1}} x_{p}^{m} I_{m-1}(2\sqrt{\rho_{2}}x_{p}) K_{0}(2x_{p}), \qquad (144)$$

where

$$x_{p}=rac{m\sqrt{1+
ho_{2}/m}}{\sqrt{\Omega}(1-
ho_{2})}\,R_{p} \ \ ext{and} \ \ \Omega=\overline{R_{p}}^{2}.$$

This form of distribution appears in certain fading problems. Its applications

will be found in the above reference. Next, the distribution of the ratio  $F = \frac{R_1^2}{\Omega_1} / \frac{R_2^2}{\Omega_2}$  takes the form

$$p(F) = \frac{F^{m-1}}{B(m,m)} \left(1 - \rho_2\right)^m (1 + F)^{-2m} \left\{1 - \frac{4\rho_2 F}{(1 + F)^2}\right\}^{-(m+\frac{1}{2})}, \quad (145)$$

where  $m_1 = m_2 = m$ . This formula is due to  $\overline{O}TA$  (1956).

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\* Ann. Conv. Record is an abbreviation for the Record at the Annual Joint Convention of Three Institutes of Electrical Engineers of Japan.

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